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THE PHASE SPACE OF 2+1 ADS GRAVITY

CARLOS SCARINCI, MSc.

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Abstract

We describe what can be called the "universal" phase space of 2+1 AdS gravity, in which the moduli spaces of globally hyperbolic AdS spacetimes with compact Cauchy surface, as well as the moduli spaces of multi black hole spacetimes are realized as submanifolds. Importantly our phase space also includes all Brown-Henneaux excitations on the conformal boundary of asymptotically AdS spacetimes, with $\mathrm{Diff}_+(\mathbb{S}^1)/\mathrm{SL}(2,\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{S}^1)/\mathrm{SL}(2,\mathbb{R})$ contained as a submanifold.

Our description of the universal phase space is obtained from results on the correspondence between maximal surfaces in AdS_3 and quasi-symmetric homeomorphisms of the unit circle. We find that the phase space can be parametrized by two copies of the universal Teichmüller space $\mathcal{T}(\mathbb{D})$, or equivalently by the cotangent bundle over $\mathcal{T}(\mathbb{D})$. This yields a symplectic map from $T^*\mathcal{T}(\mathbb{D})$ to $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ generalizing the well-known Mess map in the compact spatial surface setting.

We also relate our parametrization to the Chern-Simons formulation of 2+1 gravity and, infinitesimally, to the holographic (Fefferman-Graham) description. In particular, we relate the charges arising in the holographic description (such as the mass and angular momentum of asymptotically AdS spacetimes) to the periods of holomorphic quadratic differentials arising via the Bers embedding of $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$.

List of Papers

Much of the work presented in this dissertation has been developed in collaboration with Kirill Krasnov in

- C. Scarinci and K. Krasnov, "The universal phase space of AdS3 gravity," To appear in *Commun.Math.Phys.*, `hep-th/1111.6507`.

Chapter 6 contains results still under development in collaboration with Jean-Marc Schlenker. I have also worked on two side projects during my PhD, in collaboration with Alexander Torres-Gomez and Kirill Krasnov

- A. Torres-Gomez, K. Krasnov, and C. Scarinci, "A Unified Theory of Non-Linear Electrodynamics and Gravity," *Phys.Rev.*, **D83**, (2011), 025023, `gr-qc/1011.3641`,

and with Gianluca Delfino and Kirill Krasnov

- G. Delfino, K. Krasnov, and C. Scarinci, "Pure Connection Formalism for Gravity: Linearized Theory," `hep-th/1205.7045`.

These will not be included in the present thesis.

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Contents

Chapter 1

Introduction

1.1 Introduction to 2+1 gravity

1.1.1 The case for 2+1 gravity

Quantization of the gravitational force can be considered one of the most important open problems in theoretical physics. Almost one hundred years have passed since Einstein's publication of his general theory of relativity [1], and even more than that since the birth of quantum mechanics [2], and yet a consistent theory of quantum gravity remains elusive. There are by now many distinct approaches for tackling this problem, with some bringing remarkable developments, but so far none has produced a satisfactory formulation incorporating the lessons learned from both general relativity and quantum mechanics.

Nonetheless, one cannot overstate that the search for such a theory has been one of the main driving forces of modern theoretical physics. The development of quantum field theory, a partial marriage between relativity and quantum mechanics, has indeed produced the most successful model of fundamental interactions, the standard model of particle physics [3], bringing our understanding of nature to a level never before seen in any scientific theory. Further, one may argue that the search for quantum gravity also motivates developments in pure mathematics. Since the 80's, the close interactions between high energy physics and geometry has shed new light on fundamental problems of both disciplines. For example Donaldson's theory of 4-dimensional manifolds based on Yang-Mills gauge theory [4] and Witten's understanding of the Jones polynomial, and other low-dimensional topological invariants, using Chern-Simons [5]. In developments related to the search of quantum gravity, one has now string theory and its dualities [6, 7] playing an influential role in modern mathematical fields ranging from algebraic geometry to number theory.

The problem of quantization of gravity is clearly not easy to tackle. It presents many technical difficulties due to the highly non-linear nature of Einstein's equations. To some extent, this explains the use for ever more refined mathematical tools in the attempts made to solve it. On the other hand, there are also conceptual difficulties. For example, the importance of background structures, the role played by spacetime diffeomorphisms and the role of time [8].

General relativity in 2+1 dimensions serves as a toy model for the study of quantization of the full 3+1 dimensional theory. It presents the same conceptual issues as its higher dimensional counterpart while eliminating many of the technical problems. Its simplicity comes from absence of local degrees of freedom, implying that any solution of Einsteins equation is locally indistinguishable from either 3-dimensional Minkowski, de Sitter or anti-de Sitter spacetime, depending on the sign of the cosmological constant Λ. The theory of gravity in 2+1 dimensions is thus closely related to the study of 3-dimensional geometric structures [9], which also becomes apparent from its relation with Chern-Simons gauge theory, due to Achucarro and Townsend [10] and further developed by Witten [11]. This, in its turn, connects the theory directly to modern developments on lower dimensional topology, such as topological quantum field theories and knot invariants on 3-manifolds [5].

The Chern-Simons formulation of 2+1 gravity also provides a global understanding of the classical phase space of the theory, which is shown to be a finite dimensional cotangent bundle, allowing for the application of the usual quantization techniques. By the phase space of the theory, we mean the space of all solutions, or moduli space, on a given topological spacetime manifold, for now taken to be $\mathbb{R} \times S$ with S a closed Riemann surface. This corresponds, in Chern-Simons theory, to the space of all flat G connections over the spatial Riemann surface S modulo gauge transformation, the so called representation variety of $\pi_1(S)$ into G. Here G is the isometry group of the corresponding 3-geometry, that is, it is $ISO(2,1) \approx PSL(2,\mathbb{R}) \ltimes \mathbb{R}^3$ for $\Lambda = 0$, $SO(3,1) \approx PSL(2,\mathbb{C})$ for $\Lambda > 0$ and $SO(2,2) \approx SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$ for $\Lambda < 0$.

Gravity, however, is not equivalent to Chern-Simons theory. In fact, from the point of view of 2+1 dimensional general relativity, in spacetimes with closed spatial topology, the representation variety is too big[1]. It contains degenerate solutions with no metric interpretation and the phase space of gravity, therefore, only represents a special embedded subspace on the Chern-Simons theory phase space. In [11] Witten identifies this embedded subspace, in the case of vanishing cosmological constant, as $T^*\mathcal{T}(S)$ the cotangent bundle over Teichmüller space of an initial Cauchy surface. Moncrief [13] gives a different proof for the same description of the phase space using the canonical ADM Hamiltonian formulation in terms of space+time decomposition of the spacetime metric, thus seeing gravity as a constraint dynamical system. As could have been expected, the lack of local degrees of freedom makes it possible to explicitly solve the Hamiltonian and momentum constraints (Gauss-Codazzi equations) in terms of initial data on a Cauchy surface. Both approaches can be readily applied to other values of the cosmological

[1]Note that in more general contexts, e.g. with the inclusion of point particle singularities, the phase space of Chern-Simons theory may also be smaller than that of 2+1 gravity [12].

constant, leading to similar parametrizations of the corresponding phase spaces [14]. In this work, we are mainly interested in the case of negative cosmological constant, for which the Chern-Simons formulation also provides a parametrization by two copies of Teichmüller space $\mathcal{T}(S) \times \mathcal{T}(S)$ as shown by Mess [15].

For the uninitiated reader, Teichmüller spaces are closely related to the Riemann moduli spaces classifying all Riemann surfaces with given topology. It has several definitions as the space of conformal, complex or hyperbolic structures on 2 dimensional manifolds modulo small diffeomorphisms. Topologically it is very simple, a $6g - 6$ dimensional ball where $g \geq 2$ is the genus of the Riemann surface, but it also presents a variety of interesting structures, complex, symplectic and several distinct metric structures, including a Kählerian one. In particular, the existence of a symplectic structure allows for the discussion of quantization of such spaces, [16, 17, 18]. Accessible references can be found in [19, 20, 21, 22].

Teichmüller theory is by now a highly developed subject serving as a link between many different branches of mathematics. And, although its appearance in physically inclined studies, mainly those with lower dimensional aspects, is by no means unexpected, it may be seen as further motivation, for the mathematically inclined reader, to study 2+1 gravity. And, indeed, the close relations to the theory of 3-dimensional geometric structures and lower dimensional topology has already awaken some interest in the mathematics community, see [23, 24] and reference therein.

1.1.2 The status of 2+1 AdS gravity

Our focus on the case of negative cosmological constant is related to developments in the physical side of the story. In [25] it was shown that the theory, although “trivial” from a local point of view, admits black hole solutions with similar thermodynamical properties of their higher dimensional counterparts. This suggests the study of 2+1 gravity may indeed be very fruitful for understanding relevant aspects of more realistic quantum gravity models. One might, for example, expect that, by quantizing negative cosmological constant gravity in 2+1 dimensions, new light will be shed on the statistical nature of black hole thermodynamics.

In fact, further developments in the field seem to corroborate such expectation. In [26], a few years before the discovery of 2+1 dimensional black holes, the study of the asymptotically AdS spacetimes lead to a startling discovery that the algebra of charges associated with asymptotic symmetries is given by two copies of the centrally extended Virasoro algebra. Since upon quantization the physical states must form a representation of this algebra, this means that the quantum theory of asymptotically AdS 2+1 gravity should be a conformal field theory of the corresponding central charge. And, remarkably, the obtained value $c = 3/2G$ for this Virasoro central charge is enough, together with the assumption of modular invariance, to compute the black hole entropy in agreement with the Bekenstein-Hawking formula, see [27].

The result of [26] is now seen as a precursor of Maldacena's conjecture [28] on the AdS/CFT correspondence, considered one of the most significant development of string theory. Roughly, it states that quantum gravity or, more correctly, string theory on a anti-de Sitter background is dual to a conformally invariant supersymmetric quantum field theory on the conformal boundary of this background. Although, in the context of [26], the origin of such duality is, of course, very different from that in [28], this also serves to show how 2+1 AdS gravity relates to questions in the forefront of theoretical physics.

The AdS_3/CFT_2 picture is reasonably well-understood in the string theory setting of 3-dimensional gravity coupled to a large number of fields of string (and extra dimensional) origin. At the same time, the question of whether there really is a CFT dual to pure AdS 2+1 gravity remains open, see [29] and [30] for the most recent, yet unsuccessful, attempts in this direction. In particular, the attempt [30] to construct the genus one would-be CFT partition function by summing over modular images of the partition function of pure AdS leads to discouraging conclusions. It thus appears that pure AdS 2+1 gravity either does not have enough states to account for the Bekenstein-Hawking entropy microscopically, or that the known such states cannot be consistently put together into some CFT structure.

On the other hand, it seems sensible to tackle the problem of 2+1 quantum gravity as a problem of quantization of the arising classical phase spaces. And, we have seen above, the phase spaces of spatially compact 2+1 AdS spacetimes indeed present all the necessary ingredients for the application of usual quantization techniques. They are however too simple for the CFT type description. With finite dimensional phase spaces, quantum gravity, in this context, is simply quantum mechanics.

Note however that, in the context of 2+1 black holes, the topological setting one needs to consider is that of open spatial topologies and, therefore, the relevant phase spaces are much less understood. Constructions presented in [31, 32] show that there exists a large zoo of so called multi-black holes spacetimes, among which the BTZ black hole is only the simplest example, presenting a rather arbitrary number of asymptotic regions and internal topology. In descriptions found in the literature, [33, 34, 35, 36], much like in the compact case, the geometry of multi-black holes continues to be parametrized by two hyperbolic metrics on a spatial section. The only difference is the use of Riemann surfaces with geodesic boundaries (with hyperbolic ends attached) so that, now, there are additional moduli prescribing the lengths of each boundary component with respect to each of these metrics.

It might then seem that the phase spaces of spatially non-compact 2+1 AdS gravity are again finite dimensional. However, it is clear that the geometric description of multi-black hole spacetimes is not the whole story. In fact, purely geometric descriptions neglect the most relevant aspect of asymptotically AdS spacetimes. Namely, in the presence of conformal boundaries, the theory fails to be invariant under diffeomorphisms and new, non-geometric, degrees of freedom must be included to differentiate between inequivalent diffeomorphic configurations. And, in fact, the asymptotic symmetries of [26] consists exactly of these non-trivial diffeomorphism mapping one asymptotically AdS spacetime into a inequivalent one.

The corresponding phase spaces of 2+1 asymptotically AdS spacetimes then becomes infinite dimensional, parametrized by the space of certain diffeomorphisms of a fixed reference spacetime, and the problem of its quantization becomes much more non-trivial. It does not seem unreasonable to argue that a better description of these phase spaces may lead to a better understanding of the dual CFT picture of pure 2+1 gravity. And, although it might as well be that no such picture is possible, the development of global descriptions of these phase spaces will certainly play a relevant role for any development of a quantum theory of 2+1 AdS gravity.

1.2 The plan for this thesis

1.2.1 The universal phase space of 2+1 AdS gravity

In this thesis, we propose a new description for the phase space of 2+1 AdS spacetimes which is equally applicable in both cases of compact and non-compact spatial topologies and which, at the same time, includes the Brown-Henneaux asymptotic degrees of freedom. Our description gives a natural generalization of Mess' parametrization to non-spatially compact AdS spacetimes, following the generalization of Teichmüller space to non-compact Riemann surfaces. For simplicity let's consider the spatial topology to be that of a disc. We shall show in this thesis that the phase space of 2+1 AdS spacetimes, with topology $\mathbb{R} \times \mathbb{D}$, is given by $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$. It is important to remark that the usual definition of Teichmüller space as the space of conformal structures modulo small diffeomorphisms is not the one relevant to define $\mathcal{T}(\mathbb{D})$. Already in 2-dimensions it is too strong to impose all diffeomorphic non-compact Riemann surfaces to be considered equivalent. The most natural definition for $\mathcal{T}(\mathbb{D})$ is, rather, given by the possibility of describing the usual Teichmüller spaces of closed Riemann surfaces as quasiconformal deformation spaces of given reference surfaces. This then has a natural generalization to the unit disc and one defines the so called universal Teichmüller space, $\mathcal{T}(\mathbb{D})$, as the space of (certain equivalence classes of) quasiconformal self-maps of $\mathbb{D}$, see [21, 20, 22]. Much like the usual Teichmüller spaces of compact surfaces, the universal Teichmüller space also presents a large range of interesting structures, most importantly a symplectic structure, [37]. It is also infinite dimensional, being realized as the space of Möbius normalized quasisymmetric homeomorphisms of the unit circle, and has, therefore, enough space to accommodate the Brown-Henneaux degrees of freedom.

Interestingly, the universal Teichmüller space $\mathcal{T}(\mathbb{D})$ contains all Teichmüller spaces $\mathcal{T}(S)$ of compact Riemann surfaces as well as Teichmüller space of non-compact Riemann surfaces, now also defined as quasiconformal deformation spaces, as embedded submanifolds. This explains the adjective "universal" used in this theory. This can be seen as a consequence of the uniformization theorem of Poincaré and Kobe [38], see also [19]. Since all compact Riemann surfaces (of genus ≥ 2) are quotients of $\mathbb{D}$ by discrete groups of isometries, one can lift the quasiconformal deformations, parametrizing a Riemann surface, to a quasiconformal self-map of $\mathbb{D}$. Conversely, any quasiconformal self-map of $\mathbb{D}$ invariant under a discrete group of isometries Γ will descend to a quasiconformal map of $\mathbb{D}/\Gamma$.

It is then clear that our phase space $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ of 2+1 AdS spacetimes with topology $\mathbb{R} \times \mathbb{D}$ will then also contain all phase spaces $\mathcal{T}(S) \times \mathcal{T}(S)$, of (fixed) non-trivial spatial topology AdS spacetimes, as embedded submanifolds. In this sense, we shall call such phase space the "universal phase space".

Our construction of AdS spacetimes from points in $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ builds on and extends the results in [39] and [40]. In particular, [39] presents the existence and uniqueness of maximal surfaces in AdS_3 with prescribed boundary curve in $\partial_\infty \mathrm{AdS}_3$. For existence, only mild assumptions on the boundary curve are necessary. For uniqueness, on the other hand, further restrictions are needed. The boundary curve is then characterized as the graph of a homeomorphism of $\mathbb{S}^1$ and the uniqueness result follows if this is taken to be quasisymmetric. This then gives a one-to-one correspondence between points in $\mathcal{T}(\mathbb{D})$, now realized as quasisymmetric homeomorphisms, and maximal surfaces in AdS_3.

In [40] a closely related aspect of maximal surfaces in AdS_3 was described. Namely, it is shown that the "generalized" Gauss map from a constant mean curvature surface in AdS_3 is composed by a pair of harmonic diffeomorphisms into $\mathbb{D}$. These are not completely arbitrary, being related by the condition that their Hopf differentials add to zero. This then allows [40] to associate to a maximal surface in AdS_3 a unique minimal map between hyperbolic discs. It is also shown that the first and second fundamental forms of the maximal surface, and therefore the geometry in its domain of dependence, are completely characterized by such minimal map.

This is also related to another possible parametrization of the universal phase space, now by the cotangent bundle over universal Teichmüller space. As in the compact case, this follows from the Hamiltonian (ADM) description from the existence and uniqueness of solutions of the constraint equations, following from work of [41]. We shall see the relation between the two parametrizations is given exactly by the harmonic decomposition of minimal maps of the hyperbolic disc presented in [40], which will allow us to obtain simple expressions for the maximal surface's initial data in terms of the parametrizing point in $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$.

In the present thesis, we shall use these results, as well as other results from the universal Teichmüller literature, to describe the universal phase space in terms of quasiconformal deformations of the domain of dependence of a maximal spacelike surface in AdS_3. One non-obvious point of our construction, which is also where we depart from the works cited above, is the existence of two independent phase space directions. It might seem, from the above discussion, that a single copy of $\mathcal{T}(\mathbb{D})$ would be sufficient to describe the phase space as, indeed, it is clear that a single quasisymmetric homeomorphism suffice to describe all the initial data needed to characterize the geometry of the maximal surface's domain of dependence, via its embedding in AdS_3. One must not, however, forget that asymptotic degrees of freedom must also be taken into account.

It is clear, from the construction in [39], that the geometry of this domain of dependence is insensitive to any diffeomorphism which preserves the maximal surface's boundary curve. Such purely spatial diffeomorphism may however alter the asymptotic degrees of freedom and, therefore, cannot be considered as gauge. We thus argue in the present work that one needs not only one copy of $\mathcal{T}(\mathbb{D})$, describing the boundary curve of a maximal surface, but also a second copy, describing further quasiconformal deformations of that surface. And we shall in fact verify that these two types of deformations — the geometric corresponding to deformations of the boundary curve, and the non-geometric corresponding to quasiconformal deformations of the maximal surface it self — are canonically conjugated with respect to the symplectic structure induced by the gravitational action. Both are thus equally important as far as 2+1 AdS gravity is concerned.

1.2.2 Relations to Fefferman-Graham expansion

Another important point which needs to be made clear is that the spacetimes under discussion are not globally hyperbolic. The initial data on a spacelike surface, although enough to characterize its domain of dependence, is not enough to characterize the whole spacetime geometry. Thus, for our description of the phase space to be of any use, we need to describe a well defined analytic continuation of the spacetime metric beyond the Cauchy horizon. This will be obtained through the relation between our parametrization and another useful parametrization of asymptotically AdS spacetimes in terms of an expansion of the spacetime metric in a neighbourhood of the conformal boundary, see [42, 43, 44].

For any asymptotically AdS_3 spacetime, one can find the so called Fefferman-Graham coordinates, in a neighbourhood of (each component of) the conformal boundary, where the bulk metric takes the form

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}(\gamma_{(0)} + \rho^2\gamma_{(2)} + \rho^4\gamma_{(4)})$$

Here $\gamma_{(0)}$ is a representative of the conformal class on the conformal boundary and

$$\gamma_{(2)} = \frac{1}{2}\left(T - R_{(0)}\gamma_{(0)}\right), \quad \gamma_{(4)} = \frac{1}{4}\gamma_{(2)}\gamma_{(0)}^{-1}\gamma_{(2)}$$

with $R_{(0)}$ the Ricci scalar of $\gamma_{(0)}$ and T the Brown-York quasilocal stress-tensor [45, 46, 47]. This is a very natural description from the AdS/CFT correspondence point of view, with the Brown-York stress-tensor T being interpreted as the expectation value of the dual CFT stress-tensor [46, 47]. The components of T are, in fact, the only free parameters for asymptotic AdS metrics and thus, to some extent, also parametrize the 2+1 AdS gravity phase space. This description is, however, not entirely satisfactory. Since the Fefferman-Graham coordinate ρ extends only over a portion of spacetime near its conformal boundary, very little control over what happens inside the spacetime bulk is available. In particular, it is not possible to characterize the bulk spacetime geometry from T alone. We note, however, that the knowledge of T is very useful to compute the spacetime conserved charges.

In the present thesis, we shall also describe the relation between the universal phase space and the Fefferman-Graham description. Although only accomplished at the infinitesimal level, with an identification between the generators of quasiconformal and asymptotic deformations, this relation will be enough to demonstrate our phase space contains all Brown-Henneaux asymptotic degrees of freedom. In fact, our construction provides a new interpretation for the Brown-Henneaux generators as tangent vectors to universal Teichmüller space, which gives further justification for the naturalness of our parametrization. It will also provide us with a well defined way of analytically continuing the metric on the maximal surface's domain of dependence beyond the Cauchy horizon, and enable us to derive expressions for the spacetime charges as functions on the universal phase space. Although these expressions will only be valid at the infinitesimal level we shall see they admit natural conjectural generalizations to the finite case in terms of the so called Bers embedding of $\mathcal{T}(\mathbb{D})$.

1.2.3 The Mess map between $T^*\mathcal{T}(\mathbb{D})$ and $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$

The last point we shall address in this thesis is related to arising map Mess : $T^*\mathcal{T}(\mathbb{D}) \to \mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ from the above described parametrizations of the phase space. We note that this is still work in progress, in collaboration with Jean-Marc Schlenker, and therefore the results presented here are not yet in their final form.

We shall consider the question of whether this map is a symplectomorphism. As mentioned above, both parametrization of the phase space of AdS gravity are naturally symplectic manifolds. We shall thus obtain the relation between both these symplectic structures and the symplectic structure induced by the gravitational action. Both will be shown to agree with the induced gravitational symplectic structure coming, respectively, from the Hamiltonian and Chern-Simons formulations of 2+1 gravity. From the physical point of view this might be considered enough to claim that the Mess map is symplectic. However, from a mathematical point of view, this is perhaps not entirely justified.

We shall thus present arguments which, we believe, lead to a proof of the above claim in a more mathematically acceptable way. Having explicit expressions for the Mess map we shall compute the pull-back of the symplectic form on $\mathcal{T}(\mathbb{D})\times\mathcal{T}(\mathbb{D})$ to $T^*\mathcal{T}(\mathbb{D})$. We start by computing the derivative of the Mess map and then describe, in rather explicit terms, the Weil-Petersson symplectic structure at an arbitrary point in $\mathcal{T}(\mathbb{D})$. These ingredients are enough to compute the induced symplectic form in $T^*\mathcal{T}(S)$, which, we shall see, in fact agrees with the canonical cotangent bundle symplectic form.

There are still some subtleties in these arguments, in special with respect to boundary terms, which will not be dealt here. However, setting aside these boundary terms, the argument presented here gives a proof that the bulk contributions to the symplectic structures agree via the Mess map. Therefore, this can be seen as the first necessary steps towards a more general complete proof that the map Mess is indeed a symplectomorphism.

1.2.4 The organization of the thesis

The organization of this thesis is as follows. In the next section 1.3, we present some basic facts about the hyperbolic plane and AdS_3 spacetime with the purpose of introducing some important concepts and results needed in the course of the thesis.

In chapter 2 we shall review the construction of the phase space 2+1 gravity in the closed spatial topology case. We thus start with a quick introduction to Teichmüller theory in 2.1 and then the distinct descriptions of the the gravity phase space in the Hamiltonian and Chern-Simons formulation in section 2.2.

Chapter 3 gives some physical motivations for the constructions presented in this thesis. We review the construction of AdS spacetimes with non-compact topology in section 3.1 and then proceed to a more detailed study of the asymptotic properties of such spacetimes in section 3.2.

The construction of the universal phase space of 2+1 gravity is given in chapter 4. We starts with some review of universal Teichmüller theory in section 4.1 and its relation with maximal surfaces in AdS_3 in the following section 4.2. Section 4.3 proceeds with the construction of AdS spacetimes given pairs of points in $\mathcal{T}(\mathbb{D})$ and the equivalent construction from $T^*\mathcal{T}(\mathbb{D})$.

We then relate the universal phase space description to the, more standard, holographic description in chapter 5. Section 5.1 gives the identification between the generators of asymptotic and quasiconformal deformations and section 5.2 relates the quasilocal stress tensor with the complex analytic realization of universal Teichmüller space.

In chapter 6 we discuss the symplectic properties of the Mess map. We first relate, in section 6.1, the natural symplectic forms in both parametrizations of the universal phase space with the induced gravitational symplectic form. Then, in section 6.2, we consider the pull-back of the difference of Weil-Petersson symplectic forms in $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ to $T^*\mathcal{T}(\mathbb{D})$ showing it agrees with the canonical cotangent bundle one. We finish this section with some comments on the caveats and yet unresolved issues related to boundary contributions to the symplectic forms in consideration.

We conclude in chapter 7 with a summary of the obtained results and some discussion on future research directions.

1.3 Further preliminaries

1.3.1 The hyperbolic plane

Let $\mathbb{R}^{2,1}$ denote $\mathbb{R}^3$ with the pseudo-Riemannian flat metric of signature $(2,1)$. We define the hyperbolic plane as the hyperboloid

$$\mathbb{H}^2 = \{p = (x, y, t) \in \mathbb{R}^{2,1}; \langle p, p\rangle = -1,\, t > 0\}$$

with the induced metric, which is easily seen to be of Euclidean signature. In fact, we may introduce global coordinates on $\mathbb{H}^2$,

$$x = \sinh\chi\cos\theta, \qquad y = \sinh\chi\sin\theta, \qquad t = \cosh\chi,$$

with $(\chi, \theta) \in \mathbb{R} \times \mathbb{S}^1$, and write down the metric explicitly as

$$I_{\text{hyp}} = dx^2 + dy^2 - dt^2 = d\chi^2 + \sinh^2\chi d\theta.$$

A simple calculation show this metric has negative constant curvature -2.

Being a submanifold of Minkowski space its isometries are closely related to the signature $(2,1)$ orthogonal group. More concretely, the isometry group of $\mathbb{H}^2$ is $SO^+(2,1)$, the special orthochronous Lorentz group. This is shown isomorphic to $PSL(2,\mathbb{R}) \approx PSU(1,1)$ which becomes clear in a different parametrization via projection to the unit disc $\mathbb{D}$. We thus consider the map

$$(\chi, \theta) \to z = e^{\mathrm{i}\theta}\frac{\sinh\chi}{1 + \cosh\chi}.$$

The hyperbolic metric then acquires it usual Poincaré form

$$I_{\text{hyp}} = \frac{4|dz|^2}{(1 - |z|^2)^2} \tag{1.1}$$

whose isometries are given by Möbious transformations

$$z \to A(z) = \frac{\alpha z + \beta}{\bar{\beta}z + \bar{\alpha}}, \qquad A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in PSU(1,1).$$

An isometry $A \in PSU(1,1)$ of the hyperbolic disc is classified according to its trace as

- elliptic, if $|\mathrm{tr}\, A| < 2$;

- parabolic, if $|\mathrm{tr}\, A| = 2$;
- hyperbolic, if $|\mathrm{tr}\, A| > 2$.

This is related to the Jordan form of A. Thus, elliptic isometries are conjugated to rotations and have a single fixed point on the interior of the unit disc, parabolic isometries are conjugated to a transvection and have a single fixed point on the boundary circle, and hyperbolic isometries are conjugate to translations and have two fixed points on the boundary circle. Hyperbolic isometries are particularly important in the construction of closed Riemann surfaces, see section 2.1 for a precise definition of Riemann surface, through the uniformization theorem [38, 48] see also [19].

Theorem 1.3.1. *(Uniformization theorem) Every compact Riemann surface is conformally equivalent to*

- *the Riemann sphere $\mathbb{S}^2$, if its genus is 0;*
- *a quotient $\mathbb{C}/L$, where L is a lattice in $\mathbb{C}$, if its genus is 1;*
- *a quotient $\mathbb{D}/\Gamma$, where Γ is a discrete group of hyperbolic isometries of $\mathbb{D}$, if its genus is ≥ 2.*

We are mostly interested in the last case of hyperbolic Riemann surfaces. The discrete groups of isometries Γ appearing in the theorem are the so called (cocompact) Fuchsian groups. Taking the quotient $\mathbb{D}/\Gamma$ we are thus describing the (genus ≥ 2) Riemann surface as a fundamental domain in $\mathbb{D}$ for the action of Γ, that is, a region of $\mathbb{D}$ containing exactly one point of each orbit of Γ. Since the isometries of $\mathbb{D}$ map geodesics into each other, the fundamental domain is bounded by geodesic segments, a pair per generator of Γ. The geodesics of the hyperbolic plane, in the hyperboloid model $\mathbb{H}^2$, are obtained by intersection with planes through the origin of $\mathbb{R}^{2,1}$. In the description given by the Poincaré disc, these become arcs of circles and straight lines with end points orthogonal to the boundaries circle. Thus we may describe a (genus ≥ 2) Riemann surface as a $4g$-gon in $\mathbb{D}$ with sides identified pair wise as in figure 1.1. This can be thought as a single coordinate chart on the surface, with the only caveat of having to show unwanted boundary contributions indeed cancel by the Γ equivariance of the coordinates.

1.3.2 The anti-de Sitter spacetime

Very similar to the definition of the hyperbolic plane, we now describe the 3-dimensional anti-de Sitter spacetime. Let $\mathbb{R}^{2,2}$ denote $\mathbb{R}^4$ with the pseudo-Riemannian flat metric of signature $(2,2)$. We define the 3-dimensional anti-de-Sitter spacetime as the quadratic

$$\mathrm{AdS}_3 = \{p = (x, y, u, v) \in \mathbb{R}^{2,2}; \langle p, p\rangle = -1\}$$

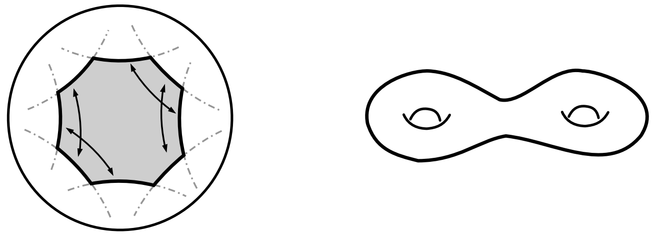

Figure 1.1: Fundamental domain for a genus 2 Riemann surface.

with the induced metric, which is now shown to have Lorentzian signature. We may again introduce global coordinates

$$x = \sinh\chi\cos\theta, \qquad y = \sinh\chi\sin\theta, \qquad u = \cosh\chi\cos t, \qquad v = \cosh\chi\sin t,$$

with $(\chi, \theta, t) \in \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$, and write down the metric explicitly as

$$g_{\mathrm{AdS}} = dx^2 + dy^2 - du^2 - dv^2 = -\cosh^2\chi dt^2 + d\chi^2 + \sinh^2\chi d\theta.$$

It is again simple to compute its curvature, $R = -6$. It is clear, in this description, that the hyperbolic plane $\mathbb{H}^2$ can be considered as a spatial section of AdS_3. This will be a useful fact for the constructions presented in this work. Note that one can also work with the universal cover of AdS_3, by unwrapping the time dimension, or with the so called projective model of AdS_3, by taking the quotient by the antipodal map. These are sometimes used as definitions of AdS_3. For our purposes, the difference introduced by considering these spaces is unimportant.

The isometry group of AdS_3 is given by $SO(2,2)_0$, which is now shown isomorphic to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$. In fact, $\mathbb{R}^{2,2}$ can be identified with the group $GL(2,\mathbb{R})$ with the metric induced by the quadratic form given by (minus) the determinant

$$p = (x, y, u, v) \in \mathbb{R}^{2,2} \quad \rightsquigarrow \quad p = \begin{pmatrix} u+x & y+v \\ y-v & u-x \end{pmatrix} \in GL(2,\mathbb{R}),$$

$$\langle p, p\rangle = -\det p = \frac{1}{2}\big(\operatorname{tr} p^2 - (\operatorname{tr} p)^2\big).$$

AdS_3 is then identified with $SL(2,\mathbb{R})$ and both $SL(2,\mathbb{R})$-actions by left and right multiplication are isometries in the connected component of the identity. This gives a surjective homomorphism $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \to SO(2,2)_0$ whose kernel is $(\pm\mathrm{Id}, \pm\mathrm{Id})$.

The geodesics and geodesic planes of AdS_3 are also obtained by intersection with hyperplanes, respectively 2 and 3-dimensional, through the origin of $\mathbb{R}^{2,2}$. In particular, AdS_3 isometries preserve geodesic and geodesic planes. This then implies, since the geodesic plane obtained by intersection with the $v = 0$ hyperplane is simply a copy of the hyperbolic plane and since AdS_3 isometries will preserve the induced metric of any embedded surfaces, that every spacelike geodesic plane in AdS_3 has a hyperbolic induced metric. In fact, choosing a new radial coordinate $r = \tanh(\chi/2) \in \mathbb{R}_+$ we obtain a description of AdS_3 where the spatial geometry is explicitly that of the hyperbolic Poincaré disc

$$g_{\mathrm{AdS}} = -\left(\frac{1+r^2}{1-r^2}\right)^2 dt^2 + \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\theta^2). \tag{1.2}$$

We shall refer to such parametrization as cylindrical since it is now possible to visualize (the universal cover of) AdS_3 as a solid cylinder.

We may also attach an asymptotic boundary to $\mathrm{AdS}_3 \approx \mathbb{D} \times \mathbb{S}^1$ by the conformally compactifying $\mathbb{D}$. We remind the reader that a (pseudo-)Riemannian (M, g) manifold is said to be conformally compact if it is diffeomorphic to the interior of a compact manifold $(\tilde{M}, \tilde{g})$ with boundary $\partial\tilde{M}$ and there exists a smooth function $\rho : \tilde{M} \to \mathbb{R}$ such that

- $\tilde{g} = \rho^2 g$ in M,
- $\rho|_M > 0$ and $\rho|_{\partial\tilde{M}} = 0$,
- $d\rho|_{\partial\tilde{M}} \neq 0$.

The spacetime $(\tilde{M}, \tilde{g})$ is then called the conformal compactification of (M, g) and $\partial\tilde{M}$, also denoted $\partial_\infty M$, is its asymptotic boundary.

Thus, in cylindrical coordinates in AdS_3, we may take

$$\rho = \frac{1-r^2}{1+r^2}, \qquad \tilde{g} = -dt^2 + \frac{4}{(1+r^2)^2}(dr^2 + r^2 d\theta^2)$$

and the asymptotic boundary $\partial_\infty \mathrm{AdS}_3 \approx \mathbb{S}^1 \times \mathbb{S}^1$ becomes a flat cylinder with induced metric

$$\gamma = -dt^2 + d\theta^2.$$

It is important to note that only the conformal structure of γ is well defined from this construction, since different choices of ρ leads to different boundary metrics on the same conformal class.

The introduction of the conformal boundary also provides an interesting identification between $SO(2,2)_0$, or rather a index 2 subgroup thereof, and $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$, see [15]. This as can be seen from the projective model of AdS_3 where one considers the projection of AdS_3 into $\mathbb{R}P^3$

$$\pi(\mathrm{AdS}_3) = \{[p] \in \mathbb{R}P^3; \langle p, p\rangle < 0\}.$$

$\pi(\mathrm{AdS}_3)$ can then be identified with $PSL(2,\mathbb{R})$ and both $PSL(2,\mathbb{R})$-actions by left and right multiplication are isometries of the induced Lorentzian structure. The conformal boundary of $\pi(\mathrm{AdS}_3)$ is the projective quadric

$$\partial\pi(\mathrm{AdS}_3) = \{[p] \in \mathbb{R}P^3; \langle p, p\rangle = 0\}.$$

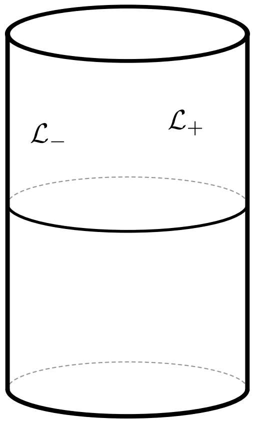

Figure 1.2: The conformal boundary of AdS_3 is foliated by two families of null geodesics.

It is foliated by two families of projective lines $\mathcal{L}_+$ and $\mathcal{L}_-$ corresponding to left and right null geodesics on the conformal boundary of AdS_3, see figure 1.2. Each line in one of the families intersects a line in the other family at a single point. In particular, we have a canonical identification of $\partial\pi(\mathrm{AdS}_3)$ with $\mathbb{R}P^1 \times \mathbb{R}P^1$ and each family $\mathcal{L}_+$ and $\mathcal{L}_-$ has a natural projective structure. Each $PSL(2,\mathbb{R})$ factor of $\mathrm{Isom}(\pi(\mathrm{AdS}_3))$ is then identified with the projective transformations of one of the families $\mathcal{L}_\pm$ and, therefore, any pair of such projective transformations uniquely determine an isometry of $\pi(\mathrm{AdS}_3)$ which can then be lifted to AdS_3. This property will be the key for relating AdS_3 geometry and Teichmüller theory in what follows. It will allow us to construct pairs of diffeomorphisms $S \to \mathbb{D}$, between any spacelike surface S and the hyperbolic disc, which will become important for constructions in chapters 2 and 4 and for the geometric interpretation of the Mess parametrization of the phase space of 2+1 dimensional globally hyperbolic AdS spacetimes with compact spatial topology.

Theorem 1.3.2. *(Mess [15]) Every (maximal) 2+1 dimensional AdS spacetime with a compact Cauchy surface is obtained as a quotient of a convex domain of* AdS_3 *by a discrete group* $\Gamma_+ \times \Gamma_-$ *of hyperbolic-hyperbolic isometries.*

Chapter 2

Compact Spatial Topology

This chapter gives a review on the description of the phase space of AdS 2+1 gravity in the spatially compact case mainly as a motivation for the constructions that follows. We start with a quick introduction to the theory of Riemann surfaces and Teichmüller theory. The main objective of section 2.1 is the definition of Teichmüller space of a given surface as the space of quasiconformal deformations of a Riemann surface structure, see [21, 22]. This will later be the starting point for the generalization to non-compact spatial topologies in chapter 4. We also present the infinitesimal description, that is, the tangent space to Teichmüller space, following [49].

We then turn to gravity with a presentation of the ADM Hamiltonian formulation in globally hyperbolic spacetimes. We start considering general dimension and cosmological constant but quickly focus on the 2+1 AdS case. With convenient choices of spatial coordinates and time foliation, we give a rather explicit description of the reduced phase of the theory as the cotangent bundle over Teichmüller space of a unique embedded maximal surface. These results can be found in [14].

Also following [14], we then describe the construction of two hyperbolic metrics on the maximal surface from its first and second fundamental forms. This produces a map from the cotangent bundle over Teichmüller space into the product of two copies of that space $T^*\mathcal{T}(S) \to \mathcal{T}(S) \times \mathcal{T}(S)$. We describe how to construct the inverse map and, therefore obtain a new parametrization of the phase space by two copies of Teichmüller space, see [15].

We finish the chapter with the interpretation of this new parametrization in terms of the Chern-Simons formulation of 2+1 gravity [11].

2.1 Teichmüller theory

2.1.1 Teichmüller spaces of closed Riemann surfaces

In this section S will denote a smooth compact oriented surface of genus ≥ 2. We shall endow S with a complex structure X, that is, an atlas of coordinate charts $z : U \to \mathbb{C}$ whose transition maps are biholomorphic. This turns S into a complex manifold and will enable us to use analytical tools in the constructions that follow. We call the pair (S, X) a Riemann surface modelled on S, see [19].

Note that the requirement of such analytical structure on the surface S is not at all a strong assumption. It is a basic fact of 2-dimensional geometry that any orientable surface admits a complex structure. In fact, given any Riemannian metric I on S, there is a canonical complex structure X_I induced by the isothermal coordinates of I. Given two isothermal coordinate charts for I where we may write

$$I = e^{2\varphi}(dx^2 + dy^2) = e^{2\tilde{\varphi}}(du^2 + dv^2),$$

it is easy to see that the transition map $(x, y) \mapsto (u, v)$ satisfy Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus, defining complex coordinates $z = x + \mathrm{i}y$, on each isothermal coordinate chart, we obtain the desired complex structure on S.

Note that the induced complex structure only depends on the conformal structure determined by I, that is, on the equivalence class $[I]$ of metrics considered up to scaling by a positive smooth function

$$\tilde{I} \in [I] \qquad \text{iff} \qquad \tilde{I} = \lambda I.$$

Conversely, a complex structure X also determines, uniquely, a conformal class of metrics on S by associating to each holomorphic coordinate patch the pull-back of the Euclidean metric on $\mathbb{C}$. Thus, in two dimensions, we have a one-to-one correspondence between complex and conformal structures on closed oriented surfaces S.

We denote the space of complex/conformal structures on S by $\mathcal{C}(S)$ and consider the natural action of the group $\mathrm{Diff}_+(S)$ of orientation preserving diffeomorphism of S on this space via pull-back: given $f \in \mathrm{Diff}_+(S)$ and a conformal structure $X \in \mathcal{C}(S)$, we may define a new conformal structure f^*X by composing the charts in X with f. The identity map $\mathrm{id} : (S, f^*X) \to (S, X)$ then becomes a biholomorphic map and the analytic structures determined by X and f^*X may not be distinguished. We would thus like to identify diffeomorphism related conformal structures and introduce a smaller space consisting of orbits of the $\mathrm{Diff}_+(S)$-action on $\mathcal{C}(S)$

$$\mathcal{M}(S) = \mathcal{C}(S)/\mathrm{Diff}_+(S).$$

This is the so called Riemann moduli space. It gives a classification of distinct conformal structures on S considered up to diffeomorphisms. Our interest in such space will be justified from our considerations about the Hamiltonian formulation of 2+1 gravity, where we shall see that 2-dimensional conformal structures represent the (physical) configurational variables in this theory.

There are several equivalent definitions of the Riemann moduli space. Importantly, it follows from the uniformization theorem (1.3.1) that the universal conformal covering of a genus ≥ 2 Riemann surface is given by the hyperbolic Poincaré disc $\mathbb{D}$, the group of deck transformations being a discrete group of isometries of $\mathbb{D}$. As seen in section 1.3 this means that every Riemann surface can be written as a quotient $(S, X) = \mathbb{D}/\Gamma$ of the Poincaré disc by a Fuchsian group Γ and, therefore, that S admits an induced complete hyperbolic metric compatible with its conformal structure. This is to say that one can always find a unique hyperbolic representative of the conformal class of metrics on (S, X). The Riemann moduli space can then be described as

$$\mathcal{M}(S) = \mathcal{H}(S)/\mathrm{Diff}_+(S),$$

where $\mathcal{H}(S)$ is the space of hyperbolic metrics on S and the action of $\mathrm{Diff}_+(S)$ is again given via pull-back.

Also, since the Fuchsian group Γ is isomorphic to the fundamental group of S, the moduli space may be described in terms of representations of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{R}) \approx PSU(1, 1)$. The equivalence relation between representations is then given by the action of the group of automorphisms of $\pi_1(S)$. More precisely, we consider the quotient space

$$\mathrm{Hom}(\pi_1(S), PSU(1, 1))/\mathrm{Aut}(\pi_1(S))$$

and the moduli space is obtained as a connected component therein. This is just saying that not all representations of $\pi_1(S)$ into $PSU(1, 1)$ will define a Riemann surface. In fact, to obtain completeness of the induced hyperbolic metric one needs the group Γ to act freely properly discontinuously. This then implies that each generator of Γ needs to be hyperbolic, that is, have trace grater then 2 in modulus, see section 1.3

The Riemann moduli space turns out to be quite complicated, presenting very non-trivial topology. It thus becomes interesting to introduce another, better behaved, space by strengthening the equivalence relations in the above definitions. The Teichmüller space of a surface S, denoted $\mathcal{T}(S)$, is again defined as the space of equivalence classes of conformal structures on S, where the equivalence relation is now taken with respect to the action of the subgroup $\mathrm{Diff}_0(S) \subset \mathrm{Diff}^+(S)$ of diffeomorphisms homotopic to the identity. Thus we define

$$\mathcal{T}(S) = \mathcal{C}(S)/\mathrm{Diff}_0(S) = \mathcal{H}(S)/\mathrm{Diff}_0(S).$$

In terms of representations of the fundamental group of S, this space is realized as a connected component of the so called representation variety

$$\mathrm{Hom}(\pi_1(S), PSU(1,1))/\mathrm{Inn}(\pi_1(S)).$$

Here, the subgroup $\mathrm{Inn}(\pi_1(S)) \subset \mathrm{Aut}(\pi_1(S))$ of inner automorphisms acts via conjugation. To obtain $\mathcal{T}(S)$ we again have to consider the restriction that the representations under consideration map the generators of $\pi_1(S)$ into hyperbolic element of $PSU(1,1)$.

Topologically, Teichmüller space $\mathcal{T}(S)$ is much simpler than $\mathcal{M}(S)$. It is shown to be simply connected, in fact it is homeomorphic to a $6g-6$ dimensional ball, where g is the genus of S. It can be considered as the covering space of the Riemann moduli space, with $\mathcal{M}(S)$ being obtained as a quotient by the so called mapping class group

$$\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{MCG}(S),$$

where

$$\mathrm{MCG}(S) = \mathrm{Diff}_+(S)/\mathrm{Diff}_0(S)$$

can be viewed as the group of connected components of $\mathrm{Diff}_+(S)$. Note, however, that the action of $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$ is not free, although it can be shown to be proper and discontinuous. Thus, $\mathcal{M}(S)$ presents topological singularities in the form of ramification points.

2.1.2 Quasiconformal deformation of complex structures

We shall now give yet another definition of $\mathcal{T}(S)$ as the space of quasiconformal deformations of a given Riemann surface. This will be useful latter in the generalization of the moduli space of noncompact Riemann surfaces and its relation to the phase space of 2+1 AdS spacetimes. In fact, the equivalence relation used so far in the definition of the Teichmüller space turns out to be rather weak. When dealing with the general case of open topologies, it will not be reasonable to consider all diffeomorphic Riemann surfaces as equivalent. For example, consider the complex plane $\mathbb{C}$ and the unit Poincaré disc $\mathbb{D}$. Although these are diffeomorphic surfaces, they cannot be consider equivalent from the complex analytic point of view and a good definition of Teichmüller space should be able to distinguish between them. We shall thus give another (equivalent in the compact case) definition of the Teichmüller space using tighter analytic conditions which will be convenient for our later purposes.

The key concept here will be that of quasiconformal maps between Riemann surfaces [21, 22]. These are certain generalizations of holomorphic maps in which deformations of angles are allowed in a controlled manner. More concretely, a quasiconformal map $f : (S, X) \to (S, X')$ is an orientation preserving almost everywhere differentiable homeomorphism with uniformly bounded dilatation. Geometrically, the dilatation of f, denoted D_f, can be understood by looking at the image of infinitesimal circles centred at each point of S. While in the case of a holomorphic map these images would again be infinitesimal circles, for a quasiconformal map the images become infinitesimal ellipses and $D_f(z)$ provides the ratio between their major and minor axis. The dilatation, therefore, controls the amount by which f is allowed to deform angles. In coordinates $z : U \to \mathbb{C}$ in (S, X), the dilatation of f is given by

$$D_f(z) = \frac{|\partial_z f| + |\partial_{\bar{z}} f|}{|\partial_z f| - |\partial_{\bar{z}} f|},$$

and the condition for f to be quasiconformal is

$$\sup_S \frac{|\partial_z f| + |\partial_{\bar{z}} f|}{|\partial_z f| - |\partial_{\bar{z}} f|} < K \tag{2.1}$$

for some positive $K \in \mathbb{R}$. It should be clear that any conformal map is in fact quasiconformal, with constant dilatation $D_f = 1$.

Another, more convenient, measure of how much a given map $f : (S, X) \to (S, X')$ fails to be holomorphic is given by the so called complex dilatation, or Beltrami coefficient, of f. This is defined in each coordinate chart $z : U \to \mathbb{C}$ in X by

$$\mu_z = \partial_{\bar{z}} f / \partial_z f.$$

It is then easy to show that the collection of Beltrami coefficients, in each coordinate chart, defines a (-1,1)-tensor on S

$$\mu_f = \mu_z \frac{d\bar{z}}{dz}.$$

To see this, let $z : U \to \mathbb{C}$ and $w : V \to \mathbb{C}$ be coordinate charts in X and denote

$$\mu_z = \frac{\partial_{\bar{z}}(f \circ z^{-1})}{\partial_z(f \circ z^{-1})}, \quad \mu_w = \frac{\partial_{\bar{w}}(f \circ w^{-1})}{\partial_w(f \circ w^{-1})}.$$

It is then clear that under the transition map $w \circ z^{-1}$ we have the required transformation rule

$$\mu_z = \mu_w(w \circ z^{-1}) \frac{\partial_{\bar{z}} \bar{w}}{\partial_z w}.$$

Thus, μ_f is also called the Beltrami differential of f.

In terms of its Beltrami differential, f is quasiconformal if and only if μ_f is bounded by 1. This is clear from the relation between the μ_f and D_f

$$|\mu_f| = \frac{|D_f| - 1}{|D_f| + 1}.$$

Also, f is holomorphic if and only if its Beltrami differential vanishes identically.

The advantage of working with Beltrami differentials is that we may then describe quasiconformal maps as solution of the Beltrami differential equation

$$\partial_{\bar{z}} f = \mu \partial_z f. \tag{2.2}$$

We make use of the following theorem which states that, up to post-composition by a Möbious transformation, quasiconformal self-maps of $\hat{\mathbb{C}}$ are completely determined by their Beltrami coefficients.

Theorem 2.1.1. *(Measurable Riemann mapping theorem [50]) Given a bounded measurable complex valued function μ on $\hat{\mathbb{C}}$ with $\|\mu\|_\infty < 1$ there exists a unique solution of the Beltrami equation (2.2), fixing $0, 1, \infty$, which is a quasiconformal homeomorphism. Further, the solution to (2.2) depends holomorphically on μ.*

We shall denote by $\mathrm{BD}(S, X)$ the space of Beltrami differentials on (S, X) and by $\mathrm{BD}(S, X)_1$ the ball of radius 1 with respect to the $\|\cdot\|_\infty$ norm. Given $\mu \in \mathrm{BD}(S, X)_1$ we may use the mapping theorem to construct a new complex structure X_μ on S and a quasiconformal map $(S, X) \to (S, X_\mu)$ whose Beltrami differential is exactly μ. For each coordinate chart $z : U \to \mathbb{C}$ in X, we first use theorem (2.1.1) to construct a quasiconformal map $f_z : z(U) \to f_z(z(U))$ whose Beltrami coefficient is μ_z and define then a new coordinate chart $\tilde{z} = f_z \circ z : U \to \mathbb{C}$. That these indeed define a new complex structure on S, follows from the vanishing of the Beltrami coefficients of the new transition maps $\tilde{w} \circ \tilde{z}^{-1}$. In fact, it is not hard to compute the Beltrami coefficient of a composition of quasiconformal maps. If f has Beltrami coefficient μ and g has Beltrami coefficient ν then $f \circ g$ has coefficient

$$\frac{\mu + \nu \circ f(\partial_{\bar{z}} \bar{f} / \partial_z f)}{1 + \bar{\mu}\nu \circ f(\partial_{\bar{z}} \bar{f} / \partial_z f)}. \tag{2.3}$$

This implies in particular that post- and pre-compositions with conformal maps does not alter the Beltrami coefficient and it is an easy exercise to show the Beltrami coefficients of the new transition maps $\tilde{w} \circ \tilde{z}^{-1}$ above vanishes. It also follows directly, that the identity map $\mathrm{id} : (S, X) \to (S, X_\mu)$ is now quasiconformal with Beltrami differential μ and, if $f : (S, X) \to (S, X')$ is another quasiconformal map with the same Beltrami differential μ, then $\mathrm{id} \circ f^{-1} : (S, X') \to (S, X_\mu)$ is conformal. We have therefore described a one-to-one correspondence between the space of bounded Beltrami differentials $\mathrm{BD}(S, X)_1$ on a Riemann surface (S, X) and the space $\mathrm{QC}(S, X)$ of quasiconformal "deformations" of that surface.

The connection to Teichmüller space can be made through the uniformization theorem (1.3.1) by translating quasiconformal deformations of $\mathbb{D}$ into deformations of Fuchsian groups. Thus, let $(S, X) = \mathbb{D}/\Gamma$ be the Fuchsian model of the Riemann surface (S, X). Given $\mu \in \mathrm{BD}(S, X)_1$ we may consider its lift to $\mathbb{D}$, that is, the unique Beltrami coefficient $\tilde{\mu} \in \mathrm{BD}(\mathbb{D})_1$ whose projection to the quotient $\mathbb{D}/\Gamma$ agrees with μ. Since μ is actually a $(-1, 1)$ tensor, this property amounts to the following Γ invariance of $\tilde{\mu}$

$$\tilde{\mu} \circ A \, \frac{\overline{A'}}{A'} = \tilde{\mu}, \qquad \forall \, A \in \Gamma. \tag{2.4}$$

We then describe the quasiconformal deformation determined by μ as follows. First, in order to solve Beltrami equation, we extend the coefficient $\tilde{\mu}$ to the whole complex plane as

$$\tilde{\mu}(z) = \begin{cases} \overline{\tilde{\mu}(1/\bar{z})} z^2/\bar{z}^2, & z \in \hat{\mathbb{C}} \backslash \mathbb{D}, \\ \tilde{\mu}(z), & z \in \mathbb{D}. \end{cases}$$

This is obtained by reflection around the unit circle $\mathbb{S}^1$ which ensures the solutions of Beltrami equation preserve the unit disc. Indeed, the solution f_μ satisfies

$$\overline{f_\mu(1/\bar{z})} = 1/f_\mu(z), \tag{2.5}$$

thus preserving $\mathbb{D}$, $\mathbb{D}^* = \hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ and also $\mathbb{S}^1$. The action of f_μ on $\mathbb{D}$ then conjugates Γ to a new Fuchsian group $\Gamma_\mu = f_\mu \circ \Gamma \circ f_\mu^{-1}$ and descends to a quasiconformal map $f_\mu : (S, X) \to (S, X_\mu)$ with $(S, X_\mu) = f_\mu(\mathbb{D})/\Gamma_\mu$.

We are now ready for our last definition of Teichmüller space. It consists in considering those conformal structures which lay in the quasiconformal deformation space $\mathrm{QC}(S, X)$ of a given Riemann surface structure (S, X). In other words, we strengthen the equivalence relation given above by restricting it to the subset $\mathrm{QC}(S, X) \subseteq \mathcal{C}(S)$ and define $\mathcal{T}(S, X)$ as the space of equivalence classes determined by this new relation. We thus define

$$\mathcal{T}(S, X) = \mathrm{QC}(S, X)/\sim$$

with the equivalence relation given by the existence of a conformal map between the quasiconformal deformations in question. Note that, in the compact case, this new definition is innocuous since any two homeomorphic compact Riemann surfaces are automatically quasiconformal and, therefore, $\mathrm{QC}(S, X) = \mathcal{C}(S)$.

Using the correspondence between the quasiconformal deformation space and the space of Beltrami coefficients we may then realize Teichmüller space as a quotient

$$\mathcal{T}(S) = \mathrm{BD}(S, X)_1/\sim .$$

Here, the equivalence between $\mu, \nu \in \mathrm{BD}(S, X)_1$ is obtained by looking at the corresponding solutions of Beltrami equation on $\hat{\mathbb{C}}$ with reflection symmetry as above. We then say $\mu \sim \nu$ if the composition $f_\nu \circ f_\mu^{-1}$ is trivial, that is, is just the identity map, when restricted to $\mathbb{S}^1$.

Other definitions for the equivalence relation on $\mathrm{BD}(S, X)_1$ are also possible by choosing different extensions of μ to the complement of the unit disc. In fact, we shall see in section 4.1 that a definition extending μ to vanish in $\hat{\mathbb{C}}\backslash\mathbb{D}$ will also be of interest in the study of noncompact Riemann surfaces.

2.1.3 Infinitesimal theory

The previous construction also allows for an explicit description of the holomorphic tangent space of $\mathcal{T}(S)$. We shall proceed with the description at the preferred reference point X, the generalization for arbitrary points X_μ will become clear in a more general context in chapter 4. Since we have realized Teichmüller space as a quotient of $\mathrm{BD}(S, X)_1$ we need to understand the derivative of the projection $\mathrm{BD}(S, X)_1 \to \mathcal{T}(S)$ at X. More precisely, we need to describe the kernel $N(S,X)$ of this map, since we may then define $T_{[0]}\mathcal{T}(S)$ as the vector space quotient of $\mathrm{BD}(S,X) = T_0\mathrm{BD}(S,X)_1$ by its subspace $N(S,X)$.

We denote by $\delta\mu$ an element of $\mathrm{BD}(S,X)$ thought of as a tangent vector at the origin of $\mathrm{BD}(S,X)_1$. As in the previous construction, let's lift and extend $\delta\mu$ by reflection to a tangent vector $\delta\tilde{\mu}$ at the origin of $\mathrm{BD}(\hat{\mathbb{C}})_1$. Then, the infinitesimal version of Beltrami equation is given by

$$\partial_{\bar{z}} f = t\delta\mu\partial_z f, \tag{2.6}$$

for t an infinitesimal parameter, and the corresponding solutions may be written

$$f_{t\delta\mu}(z) = z + t\delta z + O(t^2), \qquad \partial_{\bar{z}}\delta z = \delta\mu. \tag{2.7}$$

Now, we may consider the infinitesimal deformation $\Gamma_{t\delta\mu}$ of the Fuchsian group Γ uniformizing S by conjugating its elements with $f_{t\delta\mu}$

$$A_{t\delta\mu} = f_{t\delta\mu} \circ A \circ f_{-t\delta\mu}.$$

We say that $\delta\mu$ is infinitesimally trivial if it does not alter, to first order in t, the conformal structure of (S,X). More concretely, expanding the elements of $\Gamma_{t\delta\mu}$ as

$$A_{t\delta\mu} = A + t(\delta z \circ A - A'\delta z) + O(t^2),$$

an infinitesimal Beltrami coefficient $\delta\mu$ is said to be infinitesimally trivial if for every $A \in \Gamma$,

$$\delta A = \delta z \circ A - A'\delta z = 0.$$

The following lemma gives equivalent characterizations of the space $N(S,X)$ of such infinitesimally trivial Beltrami coefficients, see [49] for a proof.

Lemma 2.1.1.1. *Let $\delta\mu \in \mathrm{BD}(S,X)$ and consider δz the solution of the infinitesimal version of Beltrami equation, $\partial_{\bar{z}}\delta z = \delta\mu$, and $\delta A = \delta z \circ A - A'\delta z$ the variation of an element $A \in \Gamma$, as described above. Then the following conditions are equivalent*

1. *$\delta A = 0$ for every $A \in \Gamma$,*
2. *$\delta z = 0$ on $\partial\mathbb{D} = \mathbb{S}^1$,*
3. *$\frac{1}{2\mathrm{i}}\int_{\mathbb{D}/\Gamma} dz \wedge d\bar{z} q \delta\mu = 0$ for every $q \in \mathrm{HQD}(S,X)$.*

The holomorphic tangent space to $\mathcal{T}(S)$ at the base point X is then given by

$$T_X\mathcal{T}(S) = \mathrm{BD}(S,X)/N(S,X),$$

the so called harmonic Beltrami differentials.

The space $\mathrm{HQD}(S,X)$ in the lemma is the space of holomorphic quadratic differentials on the Riemann surface (S,X). These are (2,0)-tensors $q = q(z)dz^2$ whose coefficients are holomorphic in every coordinate charts. With respect to the Fuchsian model $(S,X) = \mathbb{D}/\Gamma$, q may be described as a holomorphic function $q : \mathbb{D} \to \mathbb{C}$ satisfying the invariance condition

$$q \circ A (A')^2 = q, \qquad \forall A \in \Gamma. \tag{2.8}$$

The lemma also makes use of the so called Weil-Petersson pairing between holomorphic quadratic differentials and Beltrami differentials

$$\langle q, \delta\mu \rangle_{WP} = \mathrm{i} \int_{\mathbb{D}/\Gamma} dz \wedge d\bar{z} q \delta\mu. \tag{2.9}$$

This then provides a duality between harmonic Beltrami differentials and holomorphic quadratic differentials, thus identifying $\mathrm{HQD}(S,X)$ and the holomorphic cotangent space to Teichmüller space

$$T^*_X \mathcal{T}(S) = \mathrm{HQD}(S,X).$$

The Hermitian inner-product

$$\langle h, q \rangle_{WP} = \frac{1}{2\mathrm{i}} \int_{\mathbb{D}/\Gamma} dz \wedge d\bar{z} (1 - |z|^2)^2 h \bar{q} \tag{2.10}$$

on $\mathrm{HQD}(S,X)$ can then be translated to infinitesimal Beltrami coefficients defining a Hermitian metric on $\mathcal{T}(S)$

$$\langle \delta\mu, \delta\nu \rangle_{WP} = \frac{1}{\mathrm{i}} \int_{\mathbb{D}/\Gamma} dz \wedge d\bar{z} \frac{2\delta\bar{\mu}\delta\nu}{(1 - |z|^2)^2}, \tag{2.11}$$

which defines the Weil-Petersson metric and symplectic form on $\mathcal{T}(S)$ as the real and imaginary parts, respectively.

2.2 The phase space of 2+1 gravity

2.2.1 Hamiltonian formulation of general relativity

General Relativity is a theory of Lorentzian metrics on a given topological spacetime manifold. Like most theories in physics it is defined by an action functional

$$S_{\mathrm{EH}}[g] = \frac{1}{2\pi} \int_M d^n x \sqrt{-g} (R - 2\Lambda), \tag{2.12}$$

the so called Einstein-Hilbert action, whose critical points describe the solutions of Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2.13}$$

Here, R and $R_{\mu\nu}$ are the scalar and Ricci curvatures of the metric g and Λ is the cosmological constant. Our conventions for the Riemann tensor and its contractions are

$$R^\rho{}_{\sigma\mu\nu} = 2\partial_{[\mu} \Gamma^\rho_{\nu]\sigma} + 2\Gamma^\rho_{\alpha[\mu} \Gamma^\alpha_{\nu]\sigma}, \qquad R_{\mu\nu} = R^\rho{}_{\mu\rho\nu},$$

and our unit conventions are $8G = 1$. Later we shall also set $\Lambda = -1$ for convenience.

We consider globally hyperbolic spacetimes whose topology is $\mathbb{R} \times S$, where $\mathbb{R}$ represents the time direction and S the spatial manifold, for now considered to be compact without boundaries. Einstein's equations (2.13) can then be considered as evolution equations for the spatial metric over time and GR becomes a constraint dynamical system for $(n-1)$-dimensional Riemannian metrics on S. The constraints are nothing but the Gauss-Codazzi equations relating the intrincic and extrinsic spatial geometries with the ambient spacetime geometry. These are n equations on the $n(n+1)/2$ for the components of the spacetime metric, but they further generate gauge transformations which also need to be taken into account. Thus we further need to gauge fix n components. While in 3+1 dimensions this gives the usual counting, $10 - 4 - 4 = 2$, of propagating degrees of freedom, in 2+1 dimensions the same counting gives zero propagating degrees of freedom, $6 - 3 - 3 = 0$. We shall see, this leads to a great simplification in analysing the phase space of this lower dimensional theory.

Note that the restriction to globally hyperbolic spacetimes is not strong from physical considerations. In fact global hyperbolicity is a very natural condition from the point of view of the causal structure of spacetimes. We shall not give here the actual definition of global hyperbolic spacetimes as details on causality conditions will not be relevant for the present work, see [51] for such a definition. We only remark that global hyperbolicity is equivalent to the existence of a foliation of spacetime by Cauchy surfaces from which the initial data description can be given. In particular, it implies the non existence of closed causal curves, in fact, not even "almost" closed causal curves, as desired from causality reasons.

Let us thus introduce such a foliation by choosing a smooth time function $t : M \to \mathbb{R}$ on the spacetime manifold. Such a choice is not canonically defined, but can be interpreted as a partial gauge fixing of the spacetime diffeomorphism freedom. For now, we shall not specify which function t we are to consider, with the only condition imposed being that its gradient, with respect to the spacetime metric, be always timelike on M.

We then fix a constant time Cauchy surface S and decompose the spacetime metric, following the ADM procedure [52], as

$$g = -N^2 dt^2 + I_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.14}$$

where I is the induced metric on S and N and $\vec{N}$ are the lapse function and shift vector, defined in terms of the extrinsic geometry of S as follows. Let n be the unit normal future pointing timelike vector field to S. Then every spacetime vector field ξ can be decomposed at a point in S into tangent and normal parts

$$\xi = (\xi + g(\xi, n)n) - g(\xi, n)n.$$

The lapse function and shift vector are then defined by the decomposition of the vector field ∂_t

as

$$\partial_t = \vec{N} + Nn. \tag{2.15}$$

These are closely related to the extrinsic curvature of S, a measure of the difference between the restriction of spacetime Levi-Civita connection to the initial surface S and its intrinsic Levi-Civita connection defined by the induced metric. Given ξ, ζ tangent vectors fields on S we may decompose the covariant derivative $\nabla_\xi \zeta$ in its tangent and normal parts as

$$\nabla_\xi \zeta = (\nabla_\xi \zeta + g(\nabla_\xi \zeta, n)n) - g(\nabla_\xi \zeta, n)n.$$

It is an easy exercise to see that the tangent part defines a torsion free connection on S compatible with the induced metric, thus it is nothing but the Levi-Civita connection of that metric. The normal part can then be written as

$$I\!I(\xi, \zeta)n = \nabla_\xi \zeta - D_\xi \zeta,$$

which defines the extrinsic curvature of S

$$I\!I(\xi, \zeta) = g(\nabla_\xi n, \zeta). \tag{2.16}$$

Using the ADM decomposition (2.14) of the spacetime metric as above, it is easy to obtain a coordinate expression for $I\!I$ in terms of the lapse and shift and the time derivative of spatial metric

$$I\!I_{ij} = \frac{1}{2N}(\dot{I}_{ij} - D_i N_j - D_j N_i). \tag{2.17}$$

The Einstein-Hilbert action can then be written as

$$S_{\mathrm{EH}} = \frac{1}{2\pi}\int_{\mathbb{R}} dt \int_S d^{n-1}x \sqrt{I} N({}^I R - 2\Lambda + I\!I_{ij} I\!I^{ij} - I\!I^2). \tag{2.18}$$

One continues with the Hamiltonian analysis by computing the canonically conjugated momenta associated to the spatial metric,

$$\Pi^{ij} = \frac{\delta \mathcal{L}}{\delta \dot{I}_{ij}} = \frac{\sqrt{I}}{2\pi}(I\!I^{ij} - I\!I\, I^{ij}), \tag{2.19}$$

solving for the “velocities” in terms of momenta

$$\dot{I}_{ij} = 2N\frac{2\pi}{\sqrt{I}}(\Pi_{ij} - \frac{1}{n-2}\Pi I_{ij}) + D_i N_j + D_j N_i,$$

and writing the action in the canonical form

$$S_{\mathrm{EH}} = \int_{\mathbb{R}} dt \int_S d^{n-1}x \Big(\Pi^{ij}\dot{I}_{ij} - NC - N_i C^i\Big) \tag{2.20}$$

Here,

$$C = \frac{2\pi}{\sqrt{I}}(\Pi^{ij}\Pi_{ij} - \frac{1}{n-2}\Pi^2) - \frac{\sqrt{I}}{2\pi}({}^I R - 2\Lambda), \qquad C^i = -2D_j \Pi^{ij}, \tag{2.21}$$

are the Hamiltonian and momentum constraints and are required to vanish by the field equations obtained from the variation of the lapse function and shift vector, which play the role of Lagrange multipliers. As previously mentioned, these constraints are nothing but the Gauss-Codazzi equations for the fundamental forms $(I, I\!I)$ of S.

We now read off from the action the gravitational Hamiltonian and the Poisson brackets

$$\begin{aligned} \mathcal{H}_{\rm GR} &= \int_S d^{n-1}x\Big(NC + N_iC^i\Big), \\ \{F,G\} &= \int_S d^{n-1}x\Big(\frac{\delta F}{\delta I_{ij}}\frac{\delta G}{\delta \Pi^{ij}} - \frac{\delta F}{\delta \Pi^{ij}}\frac{\delta G}{\delta I_{ij}}\Big) \end{aligned} \tag{2.22}$$

and compute the remaining evolution equations

$$\begin{aligned} \dot{I}_{ij} &= \{I_{ij}, \mathcal{H}_{\rm GR}\} = 2N\frac{2\pi}{\sqrt{I}}(\Pi_{ij} - \frac{1}{n-2}\Pi I_{ij}) + D_iN_j + D_jN_i, \\ \dot{\Pi}^{ij} &= \{\Pi^{ij}, \mathcal{H}_{\rm GR}\} = \frac{N}{2}\frac{2\pi}{\sqrt{I}}\Big(\Pi^{kl}\Pi_{kl} - \Pi^2\Big)I^{ij} - 2N\frac{2\pi}{\sqrt{I}}\Big(\Pi^{ik}I_{kl}\Pi^{lj} - \Pi\Pi^{ij}\Big) \\ &\quad - N\frac{\sqrt{I}}{2\pi}\Big({}^IR^{ij} - \frac{1}{2}({}^IR - 2\Lambda)I^{ij}\Big) + \frac{\sqrt{I}}{2\pi}\Big(D^iD^jN - I^{ij}D^kD_kN\Big) \\ &\quad + \sqrt{I}D_k\Big(N^k\frac{\Pi^{ij}}{\sqrt{I}}\Big) - \Pi^{ik}D_kN^j - \Pi^{jk}D_kN^i. \end{aligned} \tag{2.23}$$

2.2.2 2+1 dimensional globally hyperbolic spacetimes

In three dimensions a drastic simplification occurs which allows one to explicitly solve the constraints and completely describe the reduced phase space of the theory. The main reason for such simplification is the vanishing of Weyl tensor of any 3-metric, which implies the Riemann curvature is completely determined by the Ricci tensor

$$R_{\rho\sigma\mu\nu} = 2(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) - Rg_{\mu[\rho}g_{\sigma]\nu}. \tag{2.24}$$

Then, since Einstein's condition (2.13) equates Ricci to a multiple of the metric, one easily obtain that any Einstein 3-metric has constant sectional curvature and is therefore a 3-dimensional space form. This means that all 3-dimensional Einstein manifolds are locally isometric to either anti-de Sitter, Minkowski or de Sitter 3-spacetimes, depending on the value of the cosmological constant Λ.

Let's consider a spacetime (M, g) admitting a compact Cauchy surface S of genus ≥ 2. Then, M is globally hyperbolic, with topology $\mathbb{R} \times S$, and we may apply the Hamiltonian formulation presented above. Thus, let I denote the induced metric of a initial Cauchy surface S and $I\!I$ its extrinsic curvature. Without any loss of generality, we may introduce isothermal coordinates on the Cauchy surface and write

$$I = e^{2\varphi}|dz|^2, \quad I\!I = \frac{1}{2}(qdz^2 + \bar{q}d\bar{z}^2) + He^{2\varphi}|dz|^2. \tag{2.25}$$

In these coordinates the Gauss-Codazzi equations become quite simple: Codazzi equation simply relates the extrinsic curvature components q and H

$$\partial_{\bar{z}}q = e^{2\varphi}\partial_{\bar{z}}H \tag{2.26}$$

and the Gauss equation becomes an elliptic differential equation for the conformal factor $e^{2\varphi}$

$$4\partial_z\partial_{\bar{z}}\varphi = e^{2\varphi}(H^2 - \Lambda) - e^{-2\varphi}|q|^2. \tag{2.27}$$

Note that H is the mean curvature of the chosen Cauchy surface, that is, it is the trace of the extrinsic curvature $I\!I$ with respect to the induced metric I.

Let's now focus on the case of interest to the present work of a negative cosmological constant. What we now want, is to exploit a further simplification occurring for $\Lambda < 0$. We note, however, that the discussion that follows will remain valid for other values of Λ with only minor modifications, see [14]. The important aspect of globally hyperbolic AdS spacetimes that we shall make use here is the existence of a unique spacelike maximal surface, that is, a spacelike surface with vanishing mean curvature, $H = 0$. This property was first obtained in [53] as a consequence of the existence of a CMC time foliation of negative curvature spacetimes.

Thus, let us consider the initial data

$$I = e^{2\varphi}|dz|^2, \quad I\!I = \frac{1}{2}(qdz^2 + \bar{q}d\bar{z}^2) \tag{2.28}$$

on the maximal ($H = 0$) Cauchy surface. Then, the Codazzi equation simply imposes holomorphicity for the quadratic differential qdz^2 determined by the, now traceless, extrinsic curvature. The Gauss equation reads

$$4\partial_z\partial_{\bar{z}}\varphi = e^{2\varphi} - e^{-2\varphi}|q|^2 \tag{2.29}$$

and is shown to uniquely determine the conformal factor $e^{2\varphi}$ making $I = e^{2\varphi}|dz|^2$ a complete metric on S.

The remaining evolution equations (2.23) can then be solved with an appropriate choice of time gauge. We shall impose an equidistant foliation from the initial maximal Cauchy surface, which can achieved by setting $N = 1$ and $\vec{N} = 0$. The evolution equations become

$$\begin{gathered}\dot{I}_{ij} = \frac{\pi}{\sqrt{I}}(\Pi_{ij} - \Pi h_{ij}) \\ \dot{\Pi}^{ij} = \frac{\pi}{\sqrt{I}}\Big(\Pi^{kl}\Pi_{kl} - \Pi^2\Big)I^{ij} - \frac{4\pi}{\sqrt{I}}\Big(\Pi^{ik}I_{kl}\Pi^{lj} - \Pi\Pi^{ij}\Big) - \frac{\sqrt{I}}{2\pi}\Big({}^{I}R^{ij} - \frac{1}{2}({}^{I}R + 2)I^{ij}\Big)\end{gathered} \tag{2.30}$$

and it is easy, although somewhat lengthy, to verify that these are satisfied by

$$\begin{gathered}I_\tau = e^{2\varphi}|\cos\tau dz + \sin\tau e^{-2\varphi}\bar{q}d\bar{z}|^2, \\ \Pi_\tau = \frac{1}{\pi}\frac{\cos 2\tau(\bar{q}\partial_z^2 + q\partial_{\bar{z}}^2) + \sin 2\tau(e^{2\varphi} - e^{-2\varphi}|q|^2)\partial_z\partial_{\bar{z}}}{(\cos^2\tau e^{2\varphi} - \sin^2\tau e^{-2\varphi}|q|^2)}.\end{gathered} \tag{2.31}$$

The spacetime metric, in these coordinate, can then be written in a nice explicit form

$$g = -d\tau^2 + \cos^2\tau e^{2\varphi}|dz|^2 + \sin\tau\cos\tau(qdz^2 + \bar{q}d\bar{z}^2) + \sin^2\tau e^{-2\varphi}|q|^2|dz|^2. \tag{2.32}$$

Note that the choice of working with isothermal coordinates on the maximal surface also introduced a partial gauge fixing of spatial diffeomorphism freedom. The only gauge freedom we are left with is that of spatial conformal transformations and the physical configurational variable is, therefore, just the conformal structure of the maximal surface. The construction above thus describes a map from the space of globally hyperbolic AdS spacetimes with closed spatial topology to the cotangent bundle over Teichmüller space of the initial surface.

The converse statement, on the existence and uniqueness of a (maximal) globally hyperbolic AdS spacetime, with topology $M = \mathbb{R} \times S$, given a point in $T^*_{[\mu]}\mathcal{T}(S)$ follows from the application of an AdS version of the fundamental theorem of surface geometry to the universal cover of S, see [14].

Theorem 2.2.1. *(Fundamental theorem of surface geometry) Given a Riemannian metric I and a symmetric bilinear form $I\!I$ on a simply connected surface $\tilde{S}$ satisfying the Gauss-Codazzi equations (2.21), there exists a unique immersion of $\tilde{S}$ in* AdS_3 *such that the induced metric is I and the second fundamental form is $I\!I$.*

We have therefore obtained a parametrization of the reduced (physical) phase space of AdS 2+1 gravity, on a spacetime $M = \mathbb{R} \times S$, by the cotangent bundle $T^*\mathcal{T}(S)$ over Teichmüller space of the maximal Cauchy surface.

2.2.3 Mess' parametrization

Another parametrization is possible for the reduced phase space, now by two copies $\mathcal{T}(S) \times \mathcal{T}(S)$ of Teichmüller space it self. This was obtained by Mess in [15] associating pairs of flat $PSU(1,1)$ connections with hyperbolic holonomy to any Cauchy surface of a globally hyperbolic AdS spacetimes. This result is analogous to Bers' simultaneous uniformization of Riemann surfaces by quasi-Fuchsian groups [54], related to so called quasi-Fuchsian hyperbolic 3-manifolds. Mess' work, in fact, attracted attention of the mathematics community to the subject of AdS geometry by giving simple proofs of some well known theorems in low dimensional topology. See [23, 24] and references therein for recent developments associate with AdS geometry.

Given (M, g) a globally hyperbolic AdS spacetime with a closed spacelike Cauchy surface S, Mess proves that M is a quotient of AdS_3 by discrete groups of hyperbolic-hyperbolic isometries, that is, by a group of AdS_3 isometries given as the product of two cocompact hyperbolic Fuchsian groups, see theorem (1.3.2). Thus to any such globally hyperbolic AdS spacetime there is a pair of associated hyperbolic Riemann surfaces or, in other words, a point in $\mathcal{T}(S) \times \mathcal{T}(S)$. Remarkably, Mess proves that any point in $\mathcal{T}(S) \times \mathcal{T}(S)$ can be obtained from this construction and that it, in fact, completely characterizes the spacetime M. His argument is very involved and we shall follow a different, more geometric, route presented in [14] using the unique maximal surface in globally hyperbolic AdS spacetimes.

Let (M, g) be a globally hyperbolic locally AdS spacetime with a compact Cauchy surface S. Being locally AdS, the universal covering of M is AdS_3 and we may lift S to a spacelike surface $\tilde{S}$

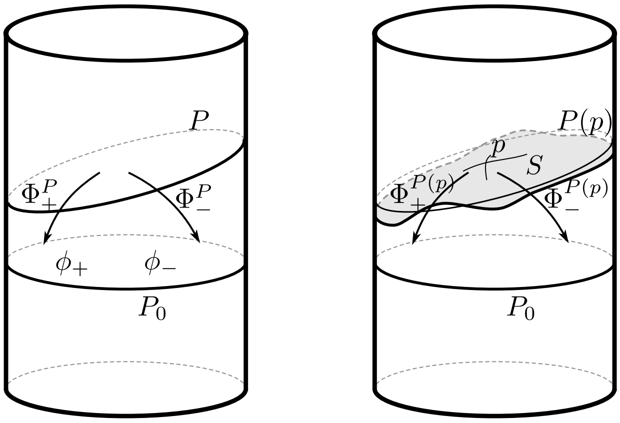

Figure 2.1: Left: AdS_3 isometries associated with a geodesic plane P. Right: Diffeomorphisms associate with spacelike surface S

therein. Associated to $\tilde{S}$ there is a pair of hyperbolic metrics on $\tilde{S}$, which we shall now define. We remind the reader, the projective model of AdS_3 provides a foliation of $\partial_\infty \mathrm{AdS}_3$ by two families of projective lines $\mathcal{L}_+$ and $\mathcal{L}_-$, corresponding to the left and right null geodesics in $\partial_\infty \mathrm{AdS}_3$, and that projective transformations on such lines completely determine AdS_3 isometries, see end of section 1.3. This will be used to construct a pair of diffeomorphisms $\Phi_\pm^{\tilde{S}} : \tilde{S} \to \mathbb{D}$ which then lead to the pair of hyperbolic metrics on $\tilde{S}$ via pull-back of the Poincaré metric on $\mathbb{D}$.

Let's start considering two geodesic spacelike surfaces P and P_0. Their conformal boundaries, $\partial_\infty P$ and $\partial_\infty P_0$ in $\partial_\infty \mathrm{AdS}_3$, intersect each line in $\mathcal{L}_+$, or rather their lifts to $\partial_\infty \mathrm{AdS}_3$, at exactly one point. We may therefore use $\mathcal{L}_+$ to define a bijection $\phi_+ : \partial_\infty P \to \partial_\infty P_0$, by simply following the lines in $\mathcal{L}_+$ from their intersection with $\partial_\infty P$ to their intersection with $\partial_\infty P_0$. This map permutes the lines of $\mathcal{L}_-$ while keeping those of $\mathcal{L}_+$ fixed and, thus, it is nothing but a projective transformation on $\mathcal{L}_-$. From the discussion of section 1.3, it then extends to a unique AdS_3 isometry sending the geodesic surface P into P_0. The same construction can be carried out using $\mathcal{L}_-$, so we obtain a pair of such isometries $\Phi_\pm^P$ mapping P to P_0, see figure 2.1.

Now, one can associate to the spacelike surface $\tilde{S}$ a pairs of diffeomorphisms $\Phi_\pm^{\tilde{S}} : \tilde{S} \to P_0$ by simply taking

$$\Phi_+^{\tilde{S}}(p) = \Phi_+^{P(p)}(p), \quad \Phi_-^{\tilde{S}}(p) = \Phi_-^{P(p)}(p),$$

where $P(p)$ is the geodesic spacelike surface tangent to $\tilde{S}$ at p, see figure 2.1. Since the induced metric I_0 on P_0 is hyperbolic, we obtain a pair of hyperbolic metrics

$$\tilde{I}_\pm = (\Phi_\pm^{\tilde{S}})^* I_0$$

and, since $\tilde{S}$ is the lift of S, these metrics then descends to a pair of hyperbolic metrics $I_\pm$ on S. Note that such pair does not depend on the choice of initial Cauchy surface and is completely characterized by the AdS spacetime. In fact, choosing a different spacelike surface S' it is not hard to see that the maps $\Phi_\pm^{\tilde{S}'} \circ (\Phi_\pm^{\tilde{S}})^{-1} : S \to S'$ are isometries between the corresponding hyperbolic metrics. We have, therefore, associated to an AdS spacetime (M, g), with compact Cauchy surface S, a point in the product $\mathcal{T}(S) \times \mathcal{T}(S)$ of two copies of the Teichmüller space of S.

The existence of a unique maximal surface now also becomes useful for the converse construction of globally hyperbolic AdS spacetimes in terms of points in $\mathcal{T}(S) \times \mathcal{T}(S)$, see [14]. First, note that the pair of hyperbolic metrics can be written in rather explicit terms, with the initial data $(I, I\!I)$ on such surface

$$I_\pm = I(E \pm JB\,\cdot\,,\, E \pm JB\,\cdot\,). \tag{2.33}$$

Here E is the identity operator on TS and J is the almost-complex structure induced by I and $B = I^{-1}I\!I$ is the shape tensor, or Weingarten operator, of the maximal surface. It is not hard to show that hyperbolicity of $I_\pm$ are a direct consequence of the Gauss-Codazzi equations (2.21) for $(I, I\!I)$.

The converse construction can now be described using the existence of minimal Lagrangian diffeomorphisms between hyperbolic surfaces $(S, I_+) \to (S, I_-)$. These are area preserving diffeomorphisms whose graph is minimal in the product $(S \times S, I_+ \times I_-)$. A useful characterization of such diffeomorphisms was given by Labourie [55], see also [14].

Lemma 2.2.1.1. *A diffeomorphism $f : (S, I_+) \to (S, I_-)$ is minimal Lagrangian if and only if there exists an operator $b : TS \to TS$, acting fiberwise, satisfying*

1. *b is self-adjoint with respect to I_+, with positive eigenvalues;*
2. *$\det b = 1$;*
3. *$d^{D^+} b = 0$, where D^+ is the Levi-Civita connection of I_+;*
4. *$f^* I_- = I_+(b\,\cdot\,, b\,\cdot\,)$.*

It is in terms of the operator b that the pair of hyperbolic metrics I_+, I_- encode the information about $(I, I\!I)$.

Thus, given a pair of points in $\mathcal{T}(S)$, represented by a pair of hyperbolic metrics $I_\pm$ on S, we consider the associated operator b satisfying conditions (1-4) above, and construct two symmetric bilinear forms on S by

$$I = \frac{1}{4} I_+(E + b\,\cdot\,, E + b\,\cdot\,), \qquad I\!I = -IJ(E + b)^{-1}(E - b) \tag{2.34}$$

which can be shown to satisfy the Gauss-Codazzi equations (2.21) as well as the tracelessness condition of $I\!I$, see [14]. Therefore, this pair describes the first and second fundamental forms of a maximal surface in a globally hyperbolic AdS spacetime and an explicit expression for the spacetime metric is then given by

$$g = -d\tau^2 + \cos^2\tau I + 2\sin\tau\cos\tau I\!I + \sin^2\tau I\!I I^{-1} I\!I. \tag{2.35}$$

again providing an efficient parametrization of globally hyperbolic AdS_3 spacetimes (with compact spatial slices) by two copies of the Teichmüller space. We shall derive a more explicit expression for the pair $(I, I\!I)$ in terms of the Beltrami representatives of $I_\pm$ in chapter 4.

We have therefore described a second parametrization of the reduced phase space of 2+1 on a spacetime $M = \mathbb{R} \times S$ by $\mathcal{T}(S) \times \mathcal{T}(S)$.

It is interesting to note that both spaces arising in the parametrizations of the reduced phase space of 2+1 AdS gravity are naturally symplectic manifold. The first, $T^*\mathcal{T}(S)$ is a finite dimensional cotangent bundle and, therefore, carries the canonical cotangent bundle symplectic structure. The second, $\mathcal{T}(S) \times \mathcal{T}(S)$, has a symplectic structure induced by the Weil-Petersson symplectic structure on each copy of $\mathcal{T}(S)$. This leads to an interesting question on whether such symplectic structures, and their possible quantization, are in any way related to 2+1 gravity. Also, since the both $T^*\mathcal{T}(S)$ and $\mathcal{T}(S) \times \mathcal{T}(S)$ parametrize the same phase space, the constructions above define a bijective map

$$\mathrm{Mess} : T^*\mathcal{T}(S) \to \mathcal{T}(S) \times \mathcal{T}(S). \tag{2.36}$$

Another question that then arises is whether this map is natural form the point of view of the symplectic structures on each space, that is, if the map Mess is a symplectomorphism between $T^*\mathcal{T}(S)$ and $\mathcal{T}(S) \times \mathcal{T}(S)$.

We shall see in chapter 6 that indeed the gravitational symplectic form, coming from the ADM formulation, agrees with the canonical cotangent bundle symplectic form. The symplectic structure on the product of Teichmüller spaces arises from a different formulation of 2+1 gravity which we now describe.

2.2.4 Chern-Simons formulation

Mess' parametrization of the reduced phase space of AdS gravity by two copies of Teichmüller space could in fact be expected from the Chern-Simons formulation of 2+1 General Relativity first introduced in [10] and further developed by Witten in [11]. In this formulation, one promotes the first order variables θ and ω, the frame field and the spin connection,to gauge fields by taking combinations

$$A^{\pm} = (\omega^a \pm \theta^a)T_a,$$

where T_a are generators of $su(1,1)$. The Einstein-Hilbert action, when written in terms of A^+ and A^-, then becomes the difference of two decoupled $PSU(1,1)$ Chern-Simons actions

$$S_{EH}[A^+, A^-] = S_{CS}[A^+] - S_{CS}[A^-]$$

where

$$S_{CS}[A] = \frac{k}{4\pi} \int_M \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{2.37}$$

In this sense we say that 2+1 GR is equivalent to Chern-Simons theory. Note, however, that the phase space of CS theory, the space of solutions modulo gauge transformations, is actually much bigger than that of GR. The critical points of the CS action are given by pairs of flat $PSU(1,1)$ connections on the spacetime manifold. Although every locally AdS metric give rise to such a pair of connections, the converse is not true. For example, the trivial connection $A^+ = A^- = 0$, a perfectly good solution of CS theory, determines a vanishing frame field and, thus, presents no metric interpretation.

The phase space of Chern-Simons theory is easily described from the Hamiltonian formalism. Thus, we choose a space+time split of spacetime $M = \mathbb{R} \times S$ and write the action as

$$S_{CS}[A] = \int dt \int_S d^2x \epsilon^{ij} \mathrm{tr} \left[A_i \partial_t A_j - A_0 \Big(\partial_i A_j - \partial_j A_i + [A_i, A_j] \Big) \right]. \tag{2.38}$$

This clearly describes a constrained dynamical system on the space of all connections over the spatial Riemann surface S, with the constraint imposing the flatness condition

$$F[A] = dA + A \wedge A = 0. \tag{2.39}$$

We further need to identify points on constraint surface which differ by a gauge transformation. In other words, not all flat connections represent physically distinct solutions and we should remove the superfluous degrees of freedom. Thus, the reduced phase space of the theory is space of flat connections modulo gauge transformations and can be realized as the $PSU(1,1)$ representation variety

$$\mathrm{Hom}(\pi_1(S), PSU(1,1))/PSU(1,1).$$

We now see, that the relation between the phase space of AdS 2+1 gravity and the phase space of $PSU(1,1) \times PSU(1,1)$ Chern-Simons theory on spacetime $M = \mathbb{R} \times S$ generalizes the description of Teichmüller space $\mathcal{T}(S)$ as a connected component of the representation variety

$$\mathrm{Hom}(\pi_1(S), PSU(1,1))/PSU(1,1)$$

mentioned in section 2.1. The phase space of Chern-Simons theory on M is nothing but two copies of the $PSU(1,1)$ representation variety for its spatial surface S and contains the 2+1 gravity phase space $\mathcal{T}(S) \times \mathcal{T}(S)$ as a connected component.

It is well know that the $PSU(1,1)$-Chern-Simons theory symplectic from reduces to the Weil-Petersson symplectic form on $T(S)$, see [56]. Thus, the Chern-Simons formulation of 2+1 gravity gives a simple argument for why one should expect the map Mess to be a symplectomorphism. We shall see this more explicitly in chapter 6.

Chapter 3

2+1 Black Holes and Holography

We now continue with the motivations for the constructions of the following chapters. In this chapter we turn to the study of locally AdS spacetimes with noncompact spatial topology. In section 3.1 we present the construction of a large class of such spacetimes, the so called multi-black holes, as quotients of AdS_3 by appropriate discrete subgroups of isometries, [31, 32]. We start considering the non-rotating case, where some simplifications occur due to the existence of a time-symmetric surface, and give the generalization to rotating black holes towards the end of the section. We also give some brief comments on their thermodynamical properties.

In section 3.2 we study, more generally, asymptotically AdS spacetimes. We shall focus on their asymptotic properties providing an explicit parametrization of these spacetimes in a neighbourhood of their conformal boundary, [42]. We shall also describe their asymptotic symmetries, following [26]. This gives an interpretation of the space of all such spacetimes as an asymptotic deformation space of a given reference spacetime, which we shall see will share similarities to the quasiconformal description of Teichmüller spaces presented in section 2.1. We also provide expressions for the charges of asymptotically AdS spacetimes and discuss their relation to the conjectured CFT description of 2+1 gravity.

3.1 Black Holes in 2+1 dimensions

3.1.1 The construction of AdS black holes

We now consider the construction of multi-black holes in 2+1 AdS gravity. As in the compact case, these are obtained from AdS_3, or some region therein, as quotients by appropriate discrete subgroups of isometries. The groups we consider are given as products of a pair of hyperbolic Fuchsian groups which are now required to be non-cocompact, that is, they uniformize non-compact complete hyperbolic surfaces with infinite area. The simplest example of such a spacetime is the BTZ black hole of [25], obtained from a discrete group generated by a single hyperbolic-hyperbolic element of $PSU(1,1) \times PSU(1,1)$. Given its simplicity and importance, we shall give some special attention to this black hole, using it as an illustrative example for the constructions we present.

Following [31, 32], we shall start describing the construction of non-rotating multi-black holes. The introduction of angular momentum will be given at the end of this section as a generalization of the non-rotating case. As we shall see, non-rotating black hole spacetimes will be completely described in terms of data on a special hyperbolic spacelike surface symmetric under time reflection. Note that the spacetimes we will consider here will not be globally hyperbolic due to the presence of a timelike conformal boundary. Thus, the following "initial data" construction is not obtained, strictly speaking, by time evolution of the initial surface but from the possibility of extending to AdS_3 the action of the Fuchsian group uniformizing that surface. This can then be interpreted as performing certain analytic continuation of the spacetime metric on the initial surface's domain of dependence beyond its Cauchy horizon.

Thus, let's consider a locally AdS spacetime (M, g) with topology $M = \mathbb{R} \times S$, where S is a noncompact surface. Further let's assume that the spacetime is time symmetric, that is, there exists a choice of time function t such that the spacetime metric is invariant under the transformation $t \mapsto -t$. Taking the ADM decomposition of the metric with respect to the foliation determined by this time function, we obtain the following transformation for the lapse function, shift vector and the components of the induced spatial metric

$$N^2(-t,x) = N^2(t,x), \qquad N^i(-t,x) = -N^i(t,x), \qquad I_{ij}(-t,x) = I_{ij}(t,x).$$

As a consequence, the extrinsic curvature (2.17) transforms as

$$N(-t,x)I\!I_{ij}(-t,x) = -N(t,x)I\!I_{ij}(t,x).$$

We then see that the time symmetry surface $t = 0$ has vanishing extrinsic curvature and, via the Gauss equation (2.21), hyperbolic induced metric.

Conversely, given a noncompact hyperbolic surface $(S, X) = \mathbb{D}/\Gamma$ we may construct a time symmetric AdS spacetime whose time symmetry surface is isometric to (S, X). This is obtained by extending the action of Γ on $\mathbb{D}$ to the whole of AdS_3 by identifying the hyperbolic disc $\mathbb{D}$ with a totally geodesic plane P in AdS_3. With no loss of generality, we may choose P to be the $v = 0$ plane in the group manifold model, where points in AdS_3 are parametrized by

$$p = \begin{pmatrix} u+x & y+v \\ y-v & u-x \end{pmatrix}, \qquad \det p = 1.$$

Then, P is the time symmetry surface with respect to, say, the global AdS_3 coordinates,

$$x = \sinh\chi\cos\theta, \qquad y = \sinh\chi\sin\theta, \qquad u = \cosh\chi\cos t, \qquad v = \cosh\chi\sin t,$$

introduced in section 1.3. It is now easy to see that the subgroup of AdS_3 isometries leaving P invariant is a diagonal copy $PSU(1,1) \subset PSU(1,1) \times PSU(1,1)$ consisting of transformations $p \mapsto ApA^T$. The action of Γ thus admits a unique extension from $\mathbb{D}$ to AdS_3 obtained by embedding Γ into this diagonal subgroup

$$\Gamma \hookrightarrow PSU(1,1) \subset PSU(1,1) \times PSU(1,1).$$

Since Γ acts freely properly discontinuously as a group of AdS_3 isometries preserving P, we obtain a black hole spacetime $(M, g) = \mathrm{AdS}_3/\Gamma \times \Gamma$ with the prescribed time symmetry surface $(S, X) = \mathbb{D}/\Gamma$.

Let us illustrate this construction more explicitly with the non-rotating BTZ black hole. In this case, $\Gamma = \langle A \rangle$ is the group generated by a single hyperbolic element $A \in PSU(1,1)$. Without loss of generality, we may choose the fixed points of A to be $z = -1, 1$ so it can be parametrized (as an $PSU(1,1)$ element) as

$$A = \begin{pmatrix} 2\cosh(\pi r_+) & -2\sinh(\pi r_+) \\ -2\sinh(\pi r_+) & 2\cosh(\pi r_+) \end{pmatrix}.$$

The relation between the $SL(2,\mathbb{R})$ model coordinates on the $P = \{v = 0\}$ and usual the complex coordinate z on $\mathbb{D}$ is given by

$$z = \frac{x + \mathrm{i}y}{1 + u}$$

so the action of A

$$z \longmapsto A(z) = \frac{\cosh(\pi r_+)z - \sinh(\pi r_+)}{-\sinh(\pi r_+)z + \cosh(\pi r_+)}$$

translates to

$$p = \begin{pmatrix} u + x & y \\ y & u - x \end{pmatrix} \quad \longmapsto \quad ApA^T = \begin{pmatrix} e^{2\pi r_+}(u + x) & y \\ y & e^{-2\pi r_+}(u - x) \end{pmatrix}$$

which can be immediately extended to the whole of $SL(2,\mathbb{R})$.

It is now possible to introduce natural coordinates adapted to the action of Γ from which its fundamental domain is easily visualized and the usual metric description is obtained. We start working in a Poincaré patch in AdS_3 with

$$\begin{gathered} u = r\cosh\theta, \quad x = r\sinh\theta \\ v = (r^2 - 1)^{1/2}\sinh t, \quad y = (r^2 - 1)^{1/2}\cosh t, \end{gathered} \tag{3.1}$$

with $(r, t, \theta) \in (1, \infty) \times \mathbb{R}^2$, so the AdS_3 metric becomes

$$g_{\mathrm{AdS}_3} = -(r^2 - 1)dt^2 + \frac{1}{r^2 - 1}dr^2 + r^2 d\theta^2. \tag{3.2}$$

The action of Γ is now simply given by translations in the coordinate θ by factors of $2\pi r_+$. Note that the level sets of the coordinate θ are given by hyperbolic geodesics on the $t = 0$

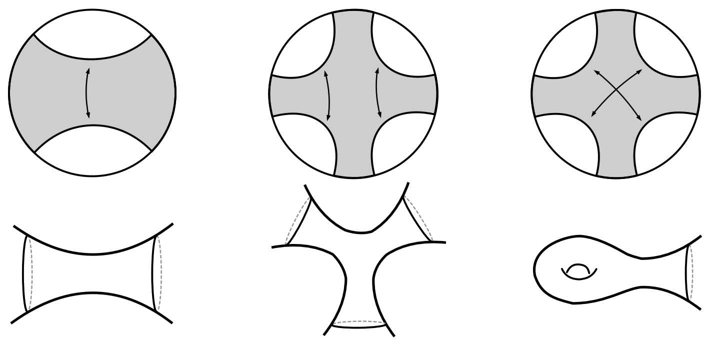

Figure 3.1: Fundamental domains of time symmetry surfaces. Left: BTZ black hole. Center: Sphere with three punctures. Right: Punctured torus.

time-symmetry surface. Thus, from the point of view of this surface, a fundamental domain of the action of Γ is given by a region between two such hyperbolic geodesics, see figure 3.1. Also, allowing t to be non-zero it is not hard to see that the level sets of θ in AdS_3 describe timelike geodesic surfaces and the fundamental domain of the action of Γ is the region between such surfaces, figure 3.2.

The next step is to consider another transformation by rescaling the coordinates as

$$t \longmapsto r_+ \tilde{t}, \qquad r \longmapsto \frac{1}{r_+}\tilde{r}, \qquad \theta \longmapsto r_+ \tilde{\theta}.$$

This ensures that, upon identification $\theta \sim \theta + 2\pi r_+$, we have a true angular coordinate $\tilde{\theta} \in [0, 2\pi]$ and leads to the usual expression for the BTZ metric

$$g_{BTZ} = -(\tilde{r}^2 - M)d\tilde{t}^2 + \frac{1}{\tilde{r}^2 - M}d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 \tag{3.3}$$

with mass parameter $M = r_+^2$. The surface $r = r_+$, where the metric becomes singular, represents the BTZ black hole's event horizon. This is only a coordinate singularity and, although the just introduced coordinates do not cover the whole of BTZ, they can be readily extended to cover a bigger region therein. We note, however, that we shall only need to consider the explicit local expression for these metrics in a neighbourhood of conformal infinity. The above coordinate patches will thus suffice for the constructions that follow.

To obtain a better global understanding of more general multi-black holes, we now describe the fundamental domain for the action of an arbitrary hyperbolic Fuchsian group Γ generalizing the previous discussion of the BTZ case. On the time-symmetry geodesic plane the fundamental domain is again obtained as the region between hyperbolic geodesic segments mapped pairwise into one another by generators of Γ, see figure 3.1. To obtain the spacetime bulk fundamental domain we then evolve these pairs of geodesics, both forward and backward in time, into pairs of geodesic surfaces. These are again mapped to one another by Γ and the region between them represents, upon identification, the multi-black hole spacetime. Note that, because of the

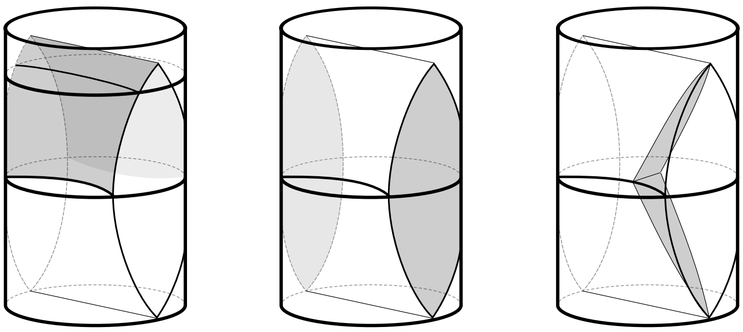

Figure 3.2: BTZ black hole. Left: Spacetime fundamental fundamental domain. Center: Boundary fundamental domain. Right: Event horizon.

positive pressure generated by the negative cosmological constant, timelike geodesics in AdS_3 are attracted to each other and, therefore, the geodesic surfaces, bounding the fundamental domain, eventually intersect and the spacetime collapses into a final singularity.

We may also include in this discussion the conformal boundary of AdS_3. We remind the reader, the isometries of AdS_3 extend to conformal transformations of $\partial_\infty \mathrm{AdS}_3$, see section 1.3. It therefore makes sense to describe the conformal boundary of quotients of AdS_3 as quotients of its conformal boundary $\partial_\infty \mathrm{AdS}_3$. We note however that the fundamental domain of the conformal boundary of a multi-black hole is not the whole of $\partial_\infty \mathrm{AdS}_3$ but only a disconnected subset therein. We again start looking at the time symmetry surface. The covering space of its conformal boundary is the circle $\mathbb{S}^1$ minus the set $\Lambda(\Gamma)$ of limit points of orbits of Γ. The complement $\mathbb{S}^1 \backslash \Lambda(\Gamma)$ is then a disjoint union of open spacelike segments on $\partial_\infty \mathrm{AdS}_3$ which are invariant under the action of Γ. Looking at the development in $\partial_\infty \mathrm{AdS}_3$ of such segments we obtain a disjoint union of open causal diamonds which are tessellated by the conformal boundaries of fundamental domains of Γ. The components of the conformal boundaries of the multi-black hole are then obtained as quotients of such diamonds by the extended action of Γ. Each of these component is then a conformal copy of $\partial_\infty \mathrm{AdS}_3$, that is, a topological cylinder with collapsed end points corresponding to the final and initial spacetime singularity. In this sense we may say that the multi-black hole spacetimes here described are asymptotically AdS [57].

Again, let's make the discussion above more explicit in the case of a BTZ spacetime. The limit set of Γ then consists only of the pair of fixed points of the generator A. Thus, the development of $\mathbb{S}^1 \backslash \Lambda(\langle A \rangle)$ is described by two causal diamonds with vertices at the fixed points of A and at the initial and final spacetime singularity, see figure 3.2.

We may write the generator A as the exponential of a Killing vector field

$$A = e^{\xi}, \qquad \xi = r_+(x\partial_u + u\partial_x).$$

The two diamonds can then be described as the regions in $\partial_\infty \mathrm{AdS}_3$ where the flow of ξ is spacelike. Introducing null coordinates $x^\pm = t \pm \theta$ on the conformal boundary of AdS_3, associated with the global cylindrical coordinates

$$x = \sinh\chi\cos\theta, \qquad y = \sinh\chi\sin\theta, \qquad u = \cosh\chi\cos t, \qquad v = \cosh\chi\sin t,$$

we may write ξ as

$$\xi = -r_+(\sin x^+ \partial_+ + \sin x^- \partial_-)$$

and it is clear from this expression the diamond regions are given by

$$-\pi < x^+ < 0, \quad 0 < x^- < \pi \qquad \text{and} \qquad 0 < x^+ < \pi, \quad -\pi < x^- < 0.$$

These are then the covering spaces of each component of the conformal boundary of the BTZ black hole. Note that the null coordinates introduced here are simply related with the global cylindrical coordinates (1.2) and not the Poincaré patch coordinates in (3.2).

To see that, upon identification, we obtain two regions asymptotic to AdS_3 we note that we may take the induced metric on $\partial_\infty \mathrm{AdS}_3$ to be

$$d\hat{s}^2 = -\frac{dx^+ dx^-}{r_+^2 \sin x^+ \sin x^-}$$

so that ξ is not only conformal Killing but now a true Killing vector field for $d\hat{s}^2$. On the BTZ quotient ξ will then be the generator of spatial rotations. There also exists a second Killing vector field orthogonal to ξ, let's call it ζ. In our coordinates this is given by

$$\zeta = -r_+(\sin x^+ \partial_+ - \sin x^- \partial_-).$$

It is easy to see that in the diamond regions above the flow of ζ is timelike so on the BTZ spacetime it will be the generator of time translations. This shows the conformal boundaries of the BTZ are nothing but timelike cylinders and we are thus entitled to call such spacetime asymptotically AdS.

Having described the fundamental domain, it is now easy to identify the event horizons of the multi-black hole. In fact, since the fundamental domain of its conformal boundary only occupy a finite region of the conformal boundary of AdS_3, there exists, in the bulk fundamental domain, a causally disconnected region from which the conformal boundary can only be reached via spacelike curves. This inner region is obtained as the complement of the causal past of the spacetime conformal boundary and the event horizons are the components of its boundary. Equivalently, the horizons can be defined as the past light cones of the final singularity of each component of the conformal boundary, figure 3.2.

We finish the discussion on the construction of AdS multi-black holes briefly describing the inclusion of angular momentum. The generalization is a natural one: instead of having a single hyperbolic Fuchsian group acting diagonally in AdS_3 we now consider the product of distinct hyperbolic Fuchsian groups $\Gamma_+ \times \Gamma_- \hookrightarrow PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R},)$ acting via

$$(A_+, A_-)p \longmapsto A_+ p A_-^T$$

on the group manifold parametrization. There will now be no surface of time symmetry so the "initial data" construction is no longer possible. The physical interpretation of these spacetimes is still available through the analysis of their conformal boundaries, in a direct generalization of the discussions above. In particular, it is possible to obtain expressions for interesting physical quantities, in terms of traces of the generators of $\Gamma_\pm$, see [32, 35].

We present here the description of the rotating BTZ black hole corresponding to a group $\langle A_+ \rangle \times \langle A_- \rangle$ generated by hyperbolic elements $A_+, A_- \in PSU(1,1)$. We may write the generators as

$$A_+ = \begin{pmatrix} e^{\pi(r_+ - r_-)} & 0 \\ 0 & e^{-\pi(r_+ - r_-)} \end{pmatrix}, \qquad A_- = \begin{pmatrix} e^{\pi(r_+ + r_-)} & 0 \\ 0 & e^{-\pi(r_+ + r_-)} \end{pmatrix}$$

and the corresponding Killing vector

$$\xi = r_+(x\partial_u + u\partial_x) + r_-(y\partial_v + v\partial_y).$$

For convenience we consider $r_+ \geq r_-$. Introducing Poincaré coordinates for AdS_3, we see that the action of Γ now also includes translations in the time coordinate by factors of $-2\pi r_-$. We thus consider more appropriate coordinates

$$t \longmapsto r_+\tilde{t} - r_-\tilde{\theta}, \qquad r^2 \longmapsto \frac{\tilde{r}^2 - r_-^2}{r_+^2 - r_-^2}, \qquad \theta \longmapsto r_+\tilde{\theta} - r_-\tilde{t}.$$

The new time coordinate is then easily seen to remain unchanged under the action of (A_+, A_-) while the angular coordinate is shifted by a factor of 2π. Upon identification $\tilde{\theta} \sim \tilde{\theta} + 2\pi$ we recover the usual expression for the rotating BTZ metric

$$g_{BTZ} = -(\tilde{r}^2 - M)d\tilde{t}^2 + \frac{1}{\tilde{r}^2 - M + \frac{J^2}{4\tilde{r}^2}} d\tilde{r}^2 - J d\tilde{t} d\tilde{\theta} + r^2 d\tilde{\theta}^2 \tag{3.4}$$

with the mass and angular momentum

$$M = r_+^2 + r_-^2, \qquad J = 2r_+r_-. \tag{3.5}$$

Note that the new coordinates just introduced are relevant for the $r \geq r_+$. Similar definitions for the appropriate coordinates in the regions $r_+ \geq r \geq r_-$ and $r_- \geq r$ are also possible leading to the same expression for the BTZ metric.

From the point of view of the conformal boundary, it is again natural to introduce null coordinates x^+, x^-. The Killing vector ξ now reads

$$\xi = -(r_+ + r_-)\sin x^+ \partial_+ - (r_+ - r_-)\sin x^- \partial_-$$

and its timelike orthogonal

$$\zeta = -(r_+ + r_-)\sin x^+ \partial_+ + (r_+ - r_-)\sin x^- \partial_- .$$

The two causal diamonds describing the covering space of the conformal boundary are still again given by

$$-\pi < x^+ < 0, \quad 0 < x^- < \pi \qquad \text{and} \qquad 0 < x^+ < \pi, \quad -\pi < x^- < 0,$$

but now the flow of ξ depends on the ratio $\Omega = r_-/r_+$. The fundamental domain of the conformal boundary of the rotating BTZ black hole is thus the same as that of the non-rotating black hole, only the identifications are different.

3.1.2 Black hole thermodynamics

The class of spacetimes described in this section, although much simpler than their 3+1 dimensional counterparts, still presents the most interesting aspects of black holes, namely their thermodynamical properties. For the general rotating BTZ black hole we obtained expressions (3.5) for mass and angular momentum in terms of the (traces of) generators (A_+, A_-). The other relevant extensive variable is the entropy which, by the Bekenstein-Hawking formula [58], is proportional to the event horizon's length

$$S = 4\pi r_+ . \tag{3.6}$$

Thus, solving for M, this gives a relation between the thermodynamical variables

$$M = \frac{S^2}{16\pi^2} + \frac{4\pi^2 J^2}{S^2}$$

from which we may compute the intensive variables. In fact, the relation above implies the following "first law"

$$dM = \frac{r_+^2 - r_-^2}{2\pi r_+} dS + \frac{r_-}{r_+} dJ. \tag{3.7}$$

Thus the BTZ black hole has temperature and angular velocity

$$T = \frac{r_+^2 - r_-^2}{2\pi r_+}, \qquad \Omega = \frac{r_-}{r_+}. \tag{3.8}$$

Although the spacetime geometry is enough to compute these thermodynamical properties of black holes, their microscopical origin remains a mystery. Only a full theory of quantum gravity will allow us to obtain a statistical description for this thermodynamical phenomenon. Nonetheless, an explanation for the origin of black hole entropy can be given from a conjectured dual CFT description of 2+1 AdS gravity.

In the present section, our considerations for the construction of multi-black hole spacetimes were mainly geometric, exploiting the lack of local bulk degrees of freedom in three dimensions. There is, however, another very important, non-geometric, aspect of multi-black holes or, more generally, asymptotically AdS spacetimes which was not taken into account. Namely, in the presence of a timelike conformal boundary the theory fails to be diffeomorphism invariant and certain class of diffeomorphisms can no longer be consider as gauge. These form the so called group of asymptotic symmetries and represent new degrees of freedom, introduced at the conformal boundary, distinguishing between inequivalent diffeomorphic configurations.

Remarkably, this plays a key role in the understanding of black hole thermodynamics. As the classical result of [26] shows, the algebra of asymptotic symmetries contains two copies of the infinite dimensional Virasoro algebra with central charge $c = 3/2G$. This thus suggests a dual description of the AdS gravitational bulk theory by a lower dimensional CFT of central charge c, in the spirit of the now acclaimed AdS/CFT correspondence of string theory. Although this is not yet enough to explain the microscopic (quantum) origin of black hole entropy, since this dual CFT remains unknown, the obtained value for the central charge c, together with modular invariance, is enough for the computation of black hole entropy in agreement with the Bekenstein-Hawking area formula (3.6), see [27, 59].

3.2 Asymptotically AdS spacetimes

3.2.1 Fefferman-Graham expansion

Let's now start with a more precise definition of asymptotically AdS spacetimes as these will be the main objects of interest in the remainder of the chapter. Since all solutions of negative cosmological constant Einstein's equation are locally indistinguishable, the key aspect we would like to impose on a spacetime so it can be considered asymptotically AdS is related to its behaviour near conformal infinity. We therefore define an asymptotically AdS spacetime to be a locally AdS spacetime (M, g) admitting a (spatial) conformal completion $(\rho, \partial_\infty M)$, see section 1.3, such that its conformal boundary $\partial_\infty M$ is a disjoint union of copies of $\mathbb{R} \times \mathbb{S}^1$ with a conformally flat Lorentzian metric [57]. From now on, we shall restrict our attention to a single component of the conformal boundary, the generalization for the other components being immediate.

A useful description of the spacetime metric can then be given in a neighbourhood of the conformal boundary. Similarly to the choice of lapse function and shift vector in the canonical formulation presented in section 2.2, one begins by fixing part of the gauge freedom by choosing a foliation of spacetime by constant radius cylinders starting from the boundary. This is achieved by taking the spatial conformal completion defining function ρ so that the spacetime metric can be written as

$$g = \frac{1}{\rho^2} d\rho^2 + \frac{1}{\rho^2}\gamma, \tag{3.9}$$

with γ the induced metric on the constant ρ cylinder. As a result of the work of Fefferman and

Graham [60], see also [42] for a physics motivated approach, Einstein's equations can then be used to determine γ order by order in ρ in an asymptotic expansion

$$\gamma = \gamma_{(0)} + \rho^2 \gamma_{(2)} + \rho^4 \gamma_{(4)} \cdots$$

starting from an arbitrary representative $\gamma_{(0)}$ of the conformal class of the boundary metric. In three dimensions this expansion stops at order ρ^4 with

$$\gamma_{(4)} = \frac{1}{4}\gamma_{(2)}\gamma_{(0)}^{-1}\gamma_{(2)}, \qquad \mathrm{tr}\,(\gamma_{(0)}^{-1}\gamma_{(2)}) = -\frac{1}{2}R_{(0)}, \qquad D_j \gamma_{(2)}{}_i^j - D_i \gamma_{(2)}{}_j^j = 0. \tag{3.10}$$

Note that a similar expansion is also possible for higher dimensional asymptotically AdS spacetimes. The possibility of writing down the Fefferman-Graham type expansion in a closed form, that is, with a finite number of terms, is, however, peculiar to 2+1 dimensions due to the absence of local degrees of freedom.

It is clear from (3.10) that not all components of the tensor $\gamma_{(2)}$ are determined by Einstein's equations. The undetermined part of $\gamma_{(2)}$, referred to as the Fefferman-Graham ambiguity [42], should then be included in the asymptotic expansion (3.9) by the introduction of an arbitrary symmetric tensor T in $\gamma_{(2)}$ satisfying

$$\gamma_{(2)} = \frac{1}{2}(T - R_{(0)}\gamma_{(0)}), \qquad \mathrm{tr}\,(\gamma_{(0)}^{-1}T) = R_{(0)}, \qquad D_j T_i^j = 0. \tag{3.11}$$

We may then further gauge fix the remaining 2-dimensional diffeomorphism freedom by working with null coordinates for a flat boundary metric

$$\gamma_{(0)} = -\frac{1}{4} dx^+ dx^-. \tag{3.12}$$

Note this is always possible by appropriately rescaling the defining function ρ. The tensor ambiguity T then becomes traceless and can be written as

$$T = a_+ (dx^+)^2 + a_- (dx^-)^2, \tag{3.13}$$

with chiral components $a_\pm$ satisfying

$$\partial_+ a_- = \partial_- a_+ = 0,$$

and the most general asymptotically AdS metric can be written as

$$g = \frac{1}{\rho^2} d\rho^2 - \frac{1}{4\rho^2} dx^+ dx^- + \frac{1}{2}(a_+ (dx^+)^2 + a_- (dx^-)^2) - \rho^2 a_+ a_- dx^+ dx^-. \tag{3.14}$$

The only freedom in this parametrization is the specification of the pair of chiral functions $a_\pm$. We have therefore obtained a realization of the phase space of asymptotically AdS spacetimes space of such pairs of functions for each conformal boundary component.

3.2.2 The quasilocal stress tensor

It remains to be shown that distinct pairs a_+, a_- indeed describe inequivalent physical configurations. This can be readily justified with a direct computation identifying the tensor ambiguity (3.13) as the quasilocal stress tensor of Brown and York [45] and, therefore, relating its components with the spacetime conserved charges.

First, note that in dealing with spatially noncompact spacetimes, the Einstein-Hilbert action (2.12) needs to be complemented by boundary terms for the well posedness of the variational principle with the given boundary conditions [61, 62]. To impose the asymptotically AdS boundary conditions we must keep fixed the conformal structure of the boundary at infinity. It is thus sufficient to add the usual York-Gibbons-Hawking integral of the boundary's mean curvature, as well as a renormalization counter term proportional to the area of the boundary for on-shell convergence [47]. Thus, we consider

$$S[g] = \frac{1}{2\pi}\int_M d^3x\sqrt{-g}(R+2) + \frac{1}{\pi}\int_{\partial M} d^2x\sqrt{-\gamma}(\Theta - 1). \tag{3.15}$$

Here, γ is the boundary metric and Θ the boundary extrinsic curvature. Note that our convention for a timelike extrinsic curvature is $\Theta_{\mu\nu} = -\nabla_{(\mu}u_{\nu)}$ with $u = \rho\partial_\rho$ the unit inwards directed normal vector field to the boundary. The quasi-local stress tensor is then obtained as the variation of this renormalized action with respect to the boundary metric

$$\frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma^{\mu\nu}} = -\frac{1}{\pi}\Big(\Theta_{\mu\nu} - \Theta^\rho{}_\rho\gamma_{\mu\nu} + \gamma_{\mu\nu}\Big) \tag{3.16}$$

in analogy with the definition of energy in Hamilton-Jacobi theory. A direct computation now shows that (3.16) is proportional to the tensor ambiguity (3.13)

$$\frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma} = \frac{1}{2\pi}\Big(a_+(dx^+)^2 + a_-(dx^-)^2\Big) = \frac{1}{2\pi}T.$$

To see this, simply denote by M_ρ the portion of the spacetime manifold interior to a constant ρ cylinder and compute the induced metric and extrinsic curvature of its timelike boundary

$$\gamma = -\frac{1}{4\rho^2}dx^+dx^- + \frac{1}{2}(a_+(dx^+)^2 + a_-(dx^-)^2) - \rho^2 a_+a_-dx^+dx^- \tag{3.17}$$

$$\Theta = -\Big(\frac{1}{4\rho^2} - \rho^2 a_+a_-\Big)dx^+dx^-. \tag{3.18}$$

The quasilocal stress tensor measures the spacetime conserved charges. For each asymptotic Killing vector field ξ we have an asymptotic charge

$$\mathcal{Q}[\xi] = \lim_{\rho\to 0}\frac{1}{\pi}\int_{\partial S_\rho} d\theta\sqrt{\sigma}n^\mu\xi^\nu T_{\mu\nu} \tag{3.19}$$

where S_ρ is the intersection of a spacelike slice $S = \{t = 0\}$ and M_ρ,

$$n = \frac{\big(\frac{1}{4\rho^2} + a_- + \rho^2a_+a_-\big)\partial_+ + \big(\frac{1}{4\rho^2} + a_+ + \rho^2a_+a_-\big)\partial_-}{\big(\frac{1}{4\rho^2} - \rho^2a_+a_-\big)\big(\frac{1}{4\rho^2} + \frac{1}{2}(a_+ + a_-) + \rho^2a_+a_-\big)^{1/2}}$$

its unit normal timelike vector field and

$$\sigma = \Big(\frac{1}{4\rho^2} + \frac{1}{2}(a_+ + a_-) + \rho^2 a_+ a_-\Big) d\theta^2$$

the induced metric on the boundary of S_ρ. We may readly compute the mass and angular momentum of the spacetime (3.14). Mass is the asymptotic charge associated with time translation symmetry ($\xi = \frac{1}{2}(\partial_+ + \partial_-)$)

$$M = \lim_{\rho\to 0} \frac{1}{2\pi} \int_{\partial S_\rho} d\theta \sqrt{\sigma} (n^+ T_{++} + n^- T_{--}) = \frac{1}{2\pi} \int_{\partial S} d\theta \, (a_+ + a_-) \tag{3.20}$$

and angular momentum the one associated with rotation symmetry ($\xi = \frac{1}{2}(\partial_+ - \partial_-)$)

$$J = \lim_{\rho\to 0} \frac{1}{2\pi} \int_{\partial \Sigma_\rho} d\theta \sqrt{\sigma} (n^+ T_{++} - n^- T_{--}) = \frac{1}{2\pi} \int_{\partial S} d\theta \, (a_+ - a_-). \tag{3.21}$$

A quick justification for these expressions can be given by computing the associated charges of the BTZ black hole. This is easily done by rewriting the BTZ metric (3.4) in Fefferman-Graham coordinates (3.14). This is achieved by introducing the coordinate

$$\frac{1}{\rho^2} = 2r^2 \left(1 - \frac{M}{2r^2} + \sqrt{1 - \frac{1}{r^2} M + \frac{J^2}{4r^4}} \right)$$

for $2r^2 > M + \sqrt{M^2 - J^2}$. We can now read the quasilocal stress tensor

$$T = \frac{1}{2}(M + J)(dx^+)^2 + \frac{1}{2}(M - J)(dx^-)^2 \tag{3.22}$$

thus supporting the above expressions for the spacetime charges. We shall give a proper justification towards the end of this section making use of the canonical formalism.

3.2.3 The group of asymptotic symmetries

It is important to note that the spacetimes described above are all related by diffeomorphisms. In fact, the gauge fixing conditions introduced earlier does not exhaust all diffeomorphism freedom. There remains a subgroup of diffeomorphisms preserving the asymptotic expansion (3.9) and the flatness of the boundary metric (3.12) but transforming the components of the quasilocal stress tensor (3.13). By the discussion above, these diffeomorphisms are not simply gauge degrees of freedom and, in fact, parametrize the phase space of asymptotically AdS spacetimes.

It is thus interesting to study the corresponding algebra of infinitesimal generators. These are easily obtained imposing invariance, under Lie derivatives, of the leading terms

$$g = \frac{1}{\rho^2} d\rho^2 - \frac{1}{4\rho^2} dx^+ dx^- + \cdots$$

of asymptotically AdS metrics. These generators, so called Brown-Henneaux vector fields, can be written in component as

$$\xi^\rho = \frac{\rho}{2}(\partial_+ \varepsilon_+ + \partial_- \varepsilon_-) + O(\rho^2), \qquad \xi^\pm = \varepsilon_\pm \mp 2\rho^2 \partial_\mp^2 \varepsilon_\mp + O(\rho^4) \tag{3.23}$$

where $\varepsilon_\pm$ are arbitrary chiral functions of $t \pm \theta$ and $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_\theta)$. The only contributions generated by such infinitesimal diffeomorphisms occurs in the components of the quasilocal stress tensor T as can be seen taking the Lie derivative of the metric (3.14)

$$\delta g = \mathcal{L}_\xi g = \frac{1}{2}(\varepsilon_+\partial_+ a_+ + 2\partial_+\varepsilon_+ a_+ - \partial_+^3\varepsilon_+)(dx^+)^2 \\ +\frac{1}{2}(\varepsilon_-\partial_- a_- + 2\partial_-\varepsilon_- a_- - \partial_-^3\varepsilon_-)(dx^-)^2 + \cdots . \tag{3.24}$$

The phase space of 2+1 asymptotically AdS spacetimes can therefore be obtained by applying Brown-Henneaux generators to a fixed "vacuum" spacetime and can be interpreted as an "asymptotic deformation space" of such reference spacetime. This is very similar to the description of the Teichmüller space as a quasiconformal deformation space of a reference Riemann surface given in section 2.1.

Note, however that the above description of the phase space is not entirely satisfactory. The Fefferman-Graham coordinate ρ only extends over a portion of the spacetime, eventually breaking down as one move towards the bulk. Thus, one only has control over what happens in a neighbourhood of the conformal boundary. In particular, it is not possible to characterize those choices of $a_\pm$ leading to particular bulk spacetime geometries.

As we shall see, this is in sharp contrast with the characterization of noncompact Riemann surfaces by quasiconformal deformations, in which the moduli (now certain homeomorphisms of the unit circle) completely determine the topological and metric properties in the whole of the Riemann surface. Given the close relation between the Teichmüller space of compact Riemann surfaces and the phase space of spatially compact AdS spacetimes presented in the last chapter, we are thus motivated to search for a better description of the phase space of asymptotically AdS spacetimes based on the Teichmüller space of noncompact Riemann surfaces.

3.2.4 Canonical formulation

Some justification is in order for our interpretation of (3.19) as the conserved charges of spacetime (3.13). Turning to the canonical formulation we shall now reobtain this above formula directly from the gravitational Hamiltonian.

In section 2.2 we have computed the Hamiltonian of general relativity ignoring all boundary contributions (2.22). Now, with a noncompact spacetime, H will need to be supplemented by boundary terms in order to become functionally differentiable. We thus write

$$H[\xi] = \int_S d^2x \xi_c^\mu C_\mu + \mathcal{Q}[\xi], \tag{3.25}$$

where C_μ are the constraints (2.21) and ξ_c^μ are surface deformation vectors corresponding to the generators of asymptotic symmetries ξ via

$$\xi_c^\perp = N\xi^t \qquad \xi_c^\rho = \xi^\rho + N^\rho\xi^t, \qquad \xi_c^\theta = \xi^\theta + N^\theta\xi^t, \tag{3.26}$$

see [26]. The boundary term $\mathcal{Q}[\xi]$ is obtained imposing the variation of $H[\xi]$ to take the form

$$\delta H[\xi] = \int_S d^2x \Big[(\cdots)^{ij} \delta h_{ij} + (\cdots)_{ij} \delta \pi^{ij} \Big].$$

Thus, to obtain the charges we need to functionally integrate the following expression

$$\begin{aligned} \delta \mathcal{Q}[\xi] = -\frac{1}{2\pi} \int d^2x \sqrt{I} D_l \Big[& (I^{ik} I^{jl} - I^{ij} I^{kl})(D_k \xi_c \delta I_{ij} - \xi_c D_k \delta I_{ij}) \\ & - \frac{4\pi}{\sqrt{I}} \xi_{ck} \delta \Pi^{kl} - \frac{2\pi}{\sqrt{I}} \xi_{ck} (2\Pi^{jl} I^{ik} - \Pi^{ij} I^{kl}) \delta I_{ij} \Big] \end{aligned} \tag{3.27}$$

Our approach will be to directly compute the asymptotic behaviour of the canonical variables and their variations in terms of the Fefferman-Graham parametrization (3.14). This computation turns out to be rather simple as we shall only need up to first subleading terms.

Thus, performing the ADM decomposition (2.14) for the $t = 0$ surface of a general asymptotically AdS spacetime (3.14) we get the following expressions for the induced spatial metric and its extrinsic curvature

$$\begin{aligned} I &= \frac{1}{\rho^2} d\rho^2 + \Big(\frac{1}{4\rho^2} + \frac{1}{2}(a_+ + a_-) + \rho^2 a_+ a_-\Big) d\theta^2, \\ \frac{\Pi}{\sqrt{I}} &= \frac{1}{\pi} \Big[\rho^5 (\partial_+ a_+ + \partial_- a_-) \partial_\rho^2 - 4\rho^4 (a_+ - a_-) \partial_\rho \partial_\theta \Big] + \cdots, \end{aligned} \tag{3.28}$$

and for their variations

$$\begin{aligned} \delta I &= \frac{1}{2}(\delta a_+ + \delta a_-) d\theta^2 + \cdots, \\ \frac{\delta \Pi}{\sqrt{I}} &= \frac{1}{\pi} \Big[\rho^5 (\partial_+ \delta a_+ + \partial_- \delta a_-) \partial_\rho^2 - 4\rho^4 (\delta a_+ - \delta a_-) \partial_\rho \partial_\theta \Big] + \cdots. \end{aligned} \tag{3.29}$$

The ellipses represent subleading terms. We remind the reader, the canonical momentum Π is not a tensor, but a tensor density weight 1. One must therefore be careful when performing its variation. In particular, this explains the square roots of the metric determinant appearing in the denominator of most formulas.

Now, using Stoke's theorem and the asymptotic relation (3.26) between the components of the vector fields ξ_c and ξ,

$$\xi_c^\perp = \frac{1}{2\rho} \xi^t + \cdots \qquad \xi_c^\rho = \xi^\rho, \qquad \xi_c^\theta = \xi^\theta + 2\rho^2 (a_+ - a_-) \xi^t + \cdots,$$

it is not hard to compute the variation of the charges at a constant ρ cylinder. In the limit $\rho \to 0$ we have

$$\delta \mathcal{Q}[\xi] = \frac{1}{2\pi} \int_{\partial S} d\theta \Big[(\delta a_+ + \delta a_-) \xi^t + (\delta a_+ - \delta a_-) \xi^\theta \Big] \tag{3.30}$$

which directly integrates to

$$\mathcal{Q}[\xi] = \frac{1}{2\pi} \int_{\partial S} d\theta \Big[(a_+ + a_-) \xi^t + (a_+ - a_-) \xi^\theta \Big] - \mathcal{Q}_0[\xi] \tag{3.31}$$

in agreement with the above expressions for mass and angular momentum. The constant term $\mathcal{Q}_0$ is determined by setting the charges of a reference spacetime to zero. This can be chosen arbitrarily but there are some obviously preferred candidates, for example the AdS spacetime it self. Our choice (3.19), however, does not take AdS_3, for which $a_+ = a_- = -1/2$, as the zero charge spacetime, but the spacetime with vanishing quasilocal stress tensor $a_+ = a_- = 0$. Although, with this convention, AdS_3 becomes negatively massive, we get the expected charges (3.5) for the BTZ black hole, see formula (3.22).

The canonical realization $\mathcal{Q}[\xi]$ of the algebra of asymptotic symmetries plays a key role in the dual CFT interpretation of asymptotically AdS gravity in 2+1 dimensions. As shown in [26] the algebra of charges contains two copies of centrally extended Virasoro algebra

$$\{\mathcal{Q}[\xi^\pm_m], \mathcal{Q}[\xi^\pm_n]\} = (m-n)\mathcal{Q}[\xi^\pm_{m+n}] + \frac{c}{12}m(m^2-1)\delta_{m+n,0}.$$

Here, $\mathcal{Q}[\xi^\pm_n]$ is the charge associated with the Brown-Henneaux generator (3.23) where $\varepsilon_\pm = e^{-\mathrm{i}n(t\pm\theta)}$, that is,

$$\xi^\pm_n = e^{-\mathrm{i}n(t\pm\theta)}(\partial_\pm - \frac{\mathrm{i}n}{2}\rho\partial_\rho) + \cdots\,, \tag{3.32}$$

Since upon quantization the quantum gravity states must form a representation of this algebra, a quantum theory of asymptotically AdS 2+1 gravity must be (dual to) a conformal field theory of the central charge c. We shall not perform here a computation of this algebra, see [26, 59] for more details, but we note that the value of the central charge can be obtained directly from the properties of the quasilocal stress tensor (3.16). In fact, we can read off the transformation of the components of T directly from the action (3.24) of the Brown-Henneaux generators on the metric (3.14)

$$\delta a_\pm = \varepsilon_\pm\partial_\pm a_\pm + 2\partial_\pm\varepsilon_\pm a_\pm - \partial^3_\pm\varepsilon_\pm. \tag{3.33}$$

This can be readily recognized as transformation of the chiral part of the stress tensor of a conformal field theory with central charge $c = 3/2G = 12$, with our choice of units $8G = 1$, see [63, 47].

Chapter 4

The Universal Phase Space

In the present chapter we give the construction of the universal phase space of 2+1 AdS gravity. We start in section 4.1 with a brief introduction to universal Teichmüller theory, generalizing the quasiconformal description of fixed compact topology Teichmüller spaces of section 2.1. The main references we shall follow are [21, 20, 22, 37]. We describe two distinct realizations, models A and B, of universal Teichmüller space $\mathcal{T}(\mathbb{D})$ which will later be related to distinct (bulk/boundary) aspects of AdS spacetimes. The relation between these models will be made very explicit at the infinitesimal level, see [49, 65], which will allow us, in chapter 5, to obtain an explicit relation between the holographic (Fefferman-Graham) description of asymptotically AdS spacetimes, presented previously, and the maximal surface description, to be introduced below.

In section 4.2 we shall quickly review the constructions of Bonsante and Schlenker [39] of maximal surfaces in AdS_3 given a point in $\mathcal{T}(\mathbb{D})$. We shall not present a complete proof of their results, but will try to give a broad idea of how these are obtained. This is mainly done for completeness and we refer the reader to [39] for more rigorous details. We shall also describe the relation between minimal Lagrangian and harmonic diffeomorphisms, associated with the generalized Gauss map of a maximal surface in AdS_3, see [40].

Section 4.3 presents the construction of AdS spacetimes from pairs of points in $\mathcal{T}(\mathbb{D})$, generalizing Mess' parametrization of spatially compact AdS spacetimes. We discuss the need for two independent Teichmüller sectors and their interpretation as “geometric” and “non-geometric” deformation directions of the domain of dependence of a geodesic surface in AdS_3. We then present a generalization of the cotangent bundle parametrization by $T^*\mathcal{T}(\mathbb{D})$ and, using the harmonic decomposition of minimal Lagrangian diffeomorphisms, give a rather simple description of a generalized Mess map $T^*\mathcal{T}(\mathbb{D}) \to \mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$.

We finish the section with a more explicit relation between the $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ parametrization and the Chern-Simons formulation of 2+1 gravity.

4.1 Universal Teichmüller theory

4.1.1 The universal Teichmüller space

In general terms, the universal Teichmüller space $\mathcal{T}(\mathbb{D})$ is the space of equivalence classes of quasiconformal deformations of the unit disc $\mathbb{D}$, generalizing the construction of Teichmüller space of closed surfaces given in section 2.1. Via the Riemann mapping theorem (2.1.1), we shall think interchangeably about bounded Beltrami coefficients and the corresponding (normalized) quasiconformal solution of Beltrami equation. Thus $\mathcal{T}(\mathbb{D})$ will be realized as a quotient of the space of bounded Beltrami coefficients on $\mathbb{D}$. For the equivalence relation, we shall present two distinct definitions, leading to two realizations of universal Teichmüller space. Although equivalent, these realizations look quite different, introducing distinct types of structures on $\mathcal{T}(\mathbb{D})$. They will also relate to distinct aspects of AdS 2+1 gravity in spatially non-compact spacetimes, as we shall see in the next sections, 4.2 and 4.3, and chapter 5.

Thus, let's consider

$$\mathrm{BD}(\mathbb{D})_1 = \left\{ \mu : \mathbb{D} \to \mathbb{C}; \|\mu\|_\infty = \sup_{\mathbb{D}} |\mu(z)| < 1 \right\},$$

the unit ball in the space of bounded Beltrami differentials on $\mathbb{D}$. As in section 2.1, we shall use the measurable Riemann mapping theorem to identify $\mathrm{BD}(\mathbb{D})_1$ and $\mathrm{QC}(\mathbb{D})$ the quasiconformal deformation space of $\mathbb{D}$. We then define the universal Teichmüller space $\mathcal{T}(\mathbb{D})$ as the space of equivalence classes of bounded Beltrami differentials $\mu \in \mathrm{BD}(\mathbb{D})_1$,

$$\mathcal{T}(\mathbb{D}) = \mathrm{BD}(\mathbb{D})_1 / \sim, \tag{4.1}$$

with the equivalence relation being defined as follows.

Model A. For each Beltrami coefficient $\mu \in \mathrm{BD}(\mathbb{D})_1$ one solves Beltrami equation

$$\partial_{\bar{z}} f = \mu \partial_z f \tag{4.2}$$

in $\mathbb{C}$ with coefficients extended to $\mathbb{D}^*$ by reflection

$$\mu(z) = \begin{cases} \mu(z), & z \in \mathbb{D}, \\ \overline{\mu(1/\bar{z})} z^2/\bar{z}^2, & z \in \mathbb{D}^*. \end{cases} \tag{4.3}$$

The solution f_μ must then satisfy

$$\overline{f_\mu(1/\bar{z})} = 1/f_\mu(z) \tag{4.4}$$

and therefore leaves invariant both $\mathbb{D}$ and $\mathbb{D}^* = \hat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ and, thus, also $\mathbb{S}^1 = \partial\mathbb{D} = \partial\mathbb{D}^*$. We then normalize such solution requiring it to fix -1, $-\mathrm{i}$ and 1 and define two bounded Beltrami coefficients μ, ν to be equivalent if the corresponding (normalized) solutions agree on $\mathbb{S}^1$

$$f_\mu\big|_{\mathbb{S}^1} = f_\nu\big|_{\mathbb{S}^1}. \tag{4.5}$$

Such boundary values of quasiconformal maps of the unit disc are homeomorphisms of the unit circle satisfying the 1-dimensional analogue of the bounded dilatation condition (2.1) of quasiconformal maps: they are allowed to alter the cross ratios of "symmetrically" placed points on $\mathbb{S}^1$ by a bounded ratio. More precisely, the so called quasisymmetric homeomorphisms are orientation preserving homeomorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$\frac{1}{k} \leq \frac{|\phi(e^{\mathrm{i}(\theta+\theta')}) - \phi(e^{\mathrm{i}\theta})|}{|\phi(e^{\mathrm{i}\theta}) - \phi(e^{\mathrm{i}(\theta-\theta')})|} \leq k, \tag{4.6}$$

for $\theta \neq \theta'$, for some constant k. We have therefore, in model A, a realization of universal Teichmüller space as the space of Möbius normalized quasisymmetric homeomorphisms of $\mathbb{S}^1$

$$\mathcal{T}(\mathbb{D}) = \mathrm{QS}(\mathbb{S}^1)/\mathrm{Möb}(\mathbb{S}^1). \tag{4.7}$$

This description readily introduces a group structure on $\mathcal{T}(\mathbb{D})$ induced by the composition of quasisymmetric homeomorphisms. The group multiplication, in terms of the Beltrami representatives, is defined as $[\lambda] = [\nu * \mu]$ if and only if the following relation is satisfied

$$(\nu \circ f_\mu) = \frac{\lambda - \mu}{1 - \lambda\bar{\mu}} \frac{\partial_z f_\mu}{\partial_{\bar{z}} \bar{f}_\mu}.$$

More explicitly, λ is the Beltrami coefficient of $f_\lambda = f_\nu \circ f_\mu$ and is given by

$$\lambda = \frac{\mu + \nu \circ f_\mu (\partial_{\bar{z}} \bar{f}_\mu / \partial_z f_\mu)}{1 + \bar{\mu}\nu \circ f_\mu (\partial_{\bar{z}} \bar{f}_\mu / \partial_z f_\mu)}. \tag{4.8}$$

Note that, in section 2.1, the equivalence relation on the quasiconformal deformation space of a closed Riemann surface $(S, X) = \mathbb{D}/\Gamma$, used to obtain the Teichmüller space $\mathcal{T}(S)$, was defined applying exactly the same procedure of model A to lifts of Beltrami differentials on (S, X). Therefore, the Teichmüller spaces $\mathcal{T}(S)$ of compact Riemann surfaces are obtained as embedded submanifold of $\mathcal{T}(\mathbb{D})$. The classes of Beltrami differentials in $\mathcal{T}(\mathbb{D})$ representing points of $\mathcal{T}(S)$ are those satisfying the Γ-invariance property (2.4)

$$\mu \circ A \, \frac{\overline{A'}}{A'} = \mu \qquad , \forall\, A \in \Gamma.$$

This, in particular, justifies the adjective "universal" used in the theory and clarifies the type of generalizations being made in this context.

Model B. Alternative definitions of equivalence relation on $\mathrm{BD}(\mathbb{D})_1$ can be introduced by choosing different extensions of the Beltrami coefficients to $\mathbb{D}^*$. We now choose a particularly natural extension setting each Beltrami coefficient to zero in $\mathbb{D}^*$

$$\mu(z) = \begin{cases} \mu(z), & z \in \mathbb{D}, \\ 0, & z \in \mathbb{D}^*. \end{cases} \tag{4.9}$$

The corresponding solution f^μ of Beltrami equation will then be holomorphic on $\mathbb{D}^*$. It is in fact biholomorphic onto its image and maps the unit disc $\mathbb{D}$ into a quasi-disc $\mathbb{D}^\mu = f^\mu(\mathbb{D})$. The solutions are now normalized to have a simple pole of residue 1 at ∞ and to satisfy $f(z) - z \to 0$ for $z \to \infty$, that is, the Möbius normalization imposes the solutions to acquire the Laurent form

$$f^\mu(z) = z\left(1 + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots\right). \tag{4.10}$$

Again, we define a pair of coefficients μ, ν to be equivalent if the corresponding normalized solutions agree now on $\mathbb{D}^*$

$$f^\mu\big|_{\mathbb{D}^*} = f^\nu\big|_{\mathbb{D}^*}. \tag{4.11}$$

Thus, in model B, we have obtained a realization of the universal Teichmüller space by Möbius normalized univalent holomorphic functions $f : \mathbb{D}^* \to \hat{\mathbb{C}}$ on the complement of the unit disc $\mathbb{D}^*$.

This realization now allows for an embedding of $\mathcal{T}(\mathbb{D})$ into the Banach space of holomorphic quadratic differentials on $\mathbb{D}^*$

$$\mathrm{HQD}(\mathbb{D}^*) = \Big\{ h : \mathbb{D}^* \to \mathbb{C} \text{ holomorphic}; |h(z)(1 - |z|^2)^2|_\infty < \infty \Big\}.$$

The associated quadratic differential to a point $[\mu] \in \mathcal{T}(\mathbb{D})$ is obtained via the Schwarzian derivative

$$\mathcal{S}(f^\mu|_{\mathbb{D}^*}) = \frac{\partial_z^3 f^\mu}{\partial_z f^\mu} - \frac{3}{2}\left(\frac{\partial_z^2 f^\mu}{\partial_z f^\mu}\right)^2 \tag{4.12}$$

of the model B solution (4.10). This is the so called Bers embedding of Teichmüller space. It provides the structure of a complex Banach manifold to universal Teichmüller space with $\mathcal{T}(\mathbb{D})$ being mapped into a bounded domain of $\mathrm{HQD}(\mathbb{D}^*)$. The complete norm of $\mathrm{HQD}(\mathbb{D}^*)$ is given by the hyperbolic sup norm

$$\|h\|_\infty = \sup_{\mathbb{D}^*} (1 - |z|^2)^2 |h(z)|,$$

see [64].

We would now like to quickly discuss the equivalence between the two descriptions above. We emphasize that the objects appearing in these two realizations are indeed very contrasting. Even the dependence of the normalized solutions f_μ and f^μ, and thus of the related quasisymmetric homeomorphisms and holomorphic quadratic differential, on the Beltrami coefficient μ are of different analytical nature. The solution f^μ in model B depends complex analytically on μ whereas for the model A solution f_μ this dependence is only real analytic. Note, also, that the group structure in $\mathcal{T}(\mathbb{D})$ becomes somewhat obscure in the B model description. The quasiconformal maps obtained in model B cannot directly be composed since they do not leave $\mathbb{D}$ and $\mathbb{D}^*$ invariant. In a similar way, the complex and Banach structures of $\mathcal{T}(\mathbb{D})$ are also not obvious from the point of view of the model A realization.

This should by no means be interpreted as a weakness of the theory. It in fact show how Teichmüller theory acts as a focus point of different branches of mathematics and provides the theory with a wider range of tools. Understanding explicitly the relation between the two models may thus prove useful in translating problems into different mathematical languages allowing them to be tackled from different angles. And, in fact, we shall argue in section 5.2 that, in the context of 2+1 gravity, this passage between the models is closely related to the conjectured dual CFT description of AdS gravity.

In the general situation, the relation between the models is not as straightforward as one would like. It is given by the so called conformal welding associating a homeomorphism of the unit circle to each Jordan region D on $\hat{\mathbb{C}}$. The first step of this construction is to use the classical Riemann mapping theorem to obtain two conformal maps $f : D^* \to \mathbb{D}$ from the complement of the Jordan region to the unit disc and $g : \mathbb{D}^* \to D$ from the region itself to the complement of the disc. Both f and g then extend continuously to the boundary ∂D and the welding homeomorphism is defined as

$$W(D) = f \circ (g\big|_{\mathbb{S}^1}) : \mathbb{S}^1 \to \mathbb{S}^1, \tag{4.13}$$

normalized to fix $-1, -\mathrm{i}, 1$. Applied to a quasi-disc D^μ, image of $\mathbb{D}^*$ under a quasiconformal univalent function f^μ from model B, the welding homeomorphism is then quasisymmetric and agrees with f_μ from model A. This can be easily seen taking $f = f_\mu \circ (f^\mu)^{-1}\big|_{(D^\mu)^*}$ and $g = f^\mu\big|_{\mathbb{D}^*}$.

As we shall see below, the relation between the models can be made much more explicit at the infinitesimal level, that is, via the derivative of the conformal welding map at the level of tangent spaces, see [65]. This explicit relation will later allow us to move back and forwards between the geometric and holographic descriptions of AdS spacetimes.

4.1.2 The tangent space to $\mathcal{T}(\mathbb{D})$

Let's consider the quotient map $\mathrm{BD}(\mathbb{D})_1 \to \mathcal{T}(\mathbb{D})$ sending each bounded Beltrami coefficient μ on $\mathbb{D}$ into its equivalence class $[\mu] \in \mathcal{T}(\mathbb{D})$. As in the compact case, see section 2.1, its derivative is a surjective map $\mathrm{BD}(\mathbb{D}) \to T_{[0]}\mathcal{T}(\mathbb{D})$ and identifies the tangent space, $T_{[0]}\mathcal{T}(\mathbb{D})$, to universal Teichmüller space at the origin with the quotient space, $\mathrm{BD}(\mathbb{D})/N(\mathbb{D})$, of the space of Beltrami coefficients by its subspace $N(\mathbb{D})$ of infinitesimally trivial coefficients, which we now wish to describe.

Denoting $\delta\mu \in \mathrm{BD}(\mathbb{D})$ a tangent vector at the origin of $\mathrm{BD}(\mathbb{D})_1$, we again look at the infinitesimal version of Beltrami equation

$$\partial_{\bar{z}} f = t\delta\mu\partial_z f \tag{4.14}$$

and the corresponding one-parameter family of infinitesimal solutions

$$f_{t\delta\mu}(z) = z + t\delta z + O(t^2), \qquad \partial_{\bar{z}}\delta z = \delta\mu. \tag{4.15}$$

Here, the extension of $\delta\mu$ to the complex plane depends on the particular model one would like to work with. For models A and B, these extensions are given by the same expressions as in the finite case (4.3) and (4.9), respectively.

We say that $\delta\mu$ is infinitesimally trivial if, to first order in t, the restriction of this family to $\mathbb{S}^1$, in model A, or to $\mathbb{D}^*$, in model B, is given just by the identity transformation, that is, if the corresponding variation $\delta z|_{\mathbb{S}^1}$ or $\delta z|_{\mathbb{D}^*}$ vanishes identically, compare this with lemma (2.1.1.1). This, of course, means that $\delta\mu$ does not change the of conformal structure on $\mathbb{D}$ and, therefore, that it is in the kernel, $N(\mathbb{D})$, of the derivative of the quotient map. As in the compact case, the infinitesimally trivial condition can be given different characterizations. It can be shown, see [21], that $\delta\mu \in N(\mathbb{D})$ is equivalent to

$$\frac{1}{2\mathrm{i}}\int_{\mathbb{D}} dz \wedge d\bar{z}\, q\delta\mu = 0,$$

for any holomorphic quadratic differential $q \in \mathrm{HQD}(\mathbb{D})$. Here, we define the space of holomorphic quadratic differentials as

$$\mathrm{HQD}(\mathbb{D}) = \left\{ q : \mathbb{D} \to \mathbb{C} \text{ holomorphic}; |q(z)(1-|z|^2)^2|_\infty < \infty \right\}.$$

We may thus write

$$N(\mathbb{D}) = \left\{ \delta\mu \in \mathrm{BD}(\mathbb{D}); \frac{1}{2\mathrm{i}}\int_{\mathbb{D}} dz \wedge d\bar{z}\, q\delta\mu = 0, \forall h \in \mathrm{HQD}(\mathbb{D}) \right\}.$$

On the other hand, any Beltrami differential, which is not infinitesimally trivial, is shown, see [21], to be dual to a holomorphic quadratic differential

$$\delta\mu = -\frac{(1-|z|^2)^2}{2}\overline{q(z)}, \tag{4.16}$$

that is, the space $\mathrm{BD}(\mathbb{D})$ can be decomposed as

$$\mathrm{BD}(\mathbb{D}) = N(\mathbb{D}) \oplus \mathrm{HBD}(\mathbb{D}),$$

with the space of harmonic Beltrami differentials on $\mathbb{D}$ being given

$$\mathrm{HBD}(\mathbb{D}) = \left\{ \delta\mu \in \mathrm{BD}(\mathbb{D});\, \delta\mu = -\frac{(1-|z|^2)^2}{2}\overline{q(z)} \text{ with } q \in \mathrm{HQD}(\mathbb{D}) \right\}.$$

We have, therefore, an identification between $T_{[0]}\mathcal{T}(\mathbb{D})$ and $\mathrm{HBD}(\mathbb{D})$.

Let us comment on the description of a tangent vector $\delta\mu$ at an arbitrary point $[\mu] \in \mathcal{T}(\mathbb{D})$. The corresponding one-parameter family of quasiconformal maps associated with $\delta\mu \in T_{[\mu]}\mathcal{T}(\mathbb{D})$ can be written

$$f_{\mu+t\delta\mu}(z) = f_\mu(z) + t\delta w \circ f_\mu(z) + O(t^2)$$

with

$$\partial_{\bar{z}}(\delta w \circ f_\mu) - \mu\partial_z(\delta w \circ f_\mu) = \delta\mu\partial_z f_\mu.$$

Note that the choice of notation $\delta w \circ f_\mu$ for the infinitesimal quasiconformal map above is convenient for we may also work in coordinates $w = f_\mu(z)$. Thus, we may compose the family of quasiconformal maps $f_{\mu+t\delta\mu}$ with f_μ^{-1} so that

$$f_{\mu+t\delta\mu} \circ f_\mu^{-1}(w) = w + t\delta w + O(t^2)$$

and δw now satisfies the previous expression for the infinitesimal Beltrami equation

$$\partial_{\bar{w}} \delta w = \delta\tilde{\mu}$$

where

$$\delta\tilde{\mu} \circ f_\mu = \frac{\delta\mu}{(1-|\mu|^2)} \frac{\partial_z f_\mu}{\partial_{\bar{z}} \bar{f}_\mu}.$$

In terms of the group structure on $\mathcal{T}(S)$ this is simply saying that the tangent vector $\delta\mu$ at $[\mu]$ is obtained from the tangent vector $\delta\tilde{\mu}$ at $[0]$ by right translation by $[\mu]$, that is,

$$f_{\mu+t\delta\mu} = f_{t\tilde{\mu}} \circ f_\mu, \qquad \mu + t\delta\mu = t\delta\tilde{\mu} * \mu = \mu + t(1-|\mu|^2)\delta\tilde{\mu} \circ f_\mu \frac{\partial_{\bar{w}} \bar{f}_\mu}{\partial_w f_\mu},$$

so we may write $T_{[\mu]}\mathcal{T}(\mathbb{D}) = (R_{[\mu]})_* T_{[0]}\mathcal{T}(\mathbb{D})$. We shall therefore continue to work at the base reference point $[0] \in \mathcal{T}(\mathbb{D})$, understanding that the following constructions are easily translated to arbitrary points with the group structure on $\mathcal{T}(\mathbb{D})$.

For the model A realization, one may think of the family $f_{t\delta\mu}$ of infinitesimal quasiconformal deformations as the one-parameter flow of a vector field $\delta z \partial_z$ on $\mathbb{D}$. Its restriction to $\mathbb{S}^1$ is then given

$$\delta z \partial_z \big|_{\mathbb{S}^1} = u(e^{\mathrm{i}\theta})\partial_\theta \tag{4.17}$$

with

$$u(e^{\mathrm{i}\theta}) = \frac{\delta z(e^{\mathrm{i}\theta})}{\mathrm{i}e^{\mathrm{i}\theta}} = \sum_{k\neq -1,0,1} u_k e^{\mathrm{i}k\theta} \tag{4.18}$$

an element of the so called Zygmund class $\Lambda(\mathbb{S}^1)$, [66, 67]. The Zygmund class is defined in the upper half plane model by

$$\Lambda(\mathbb{R}) = \Big\{ A : \mathbb{R} \to \mathbb{R} \text{ continuous such that} \\ |A(x+t) + A(x-t) - 2A(x)| \leq k|t|, k > 0 \Big\}.$$

In passing to the Poincaré disc model, we relate the real line Zygmund class functions A to the unit circle Zygmund class functions u via

$$A_u(x) = \frac{1}{2}(x^2+1)u\left(\frac{x-\mathrm{i}}{x+\mathrm{i}}\right).$$

Thus, the Zygmund class on the unit circle is defined as

$$\Lambda(\mathbb{S}^1) = \Big\{ u : \mathbb{S}^1 \to \mathbb{R} \text{ continuous such that } A_u \in \Lambda(\mathbb{R}) \Big\},$$

Note that, since u is a real function, $u_{-k} = \bar{u}_k$. Also, the coefficients u_{-1}, u_0, u_1 were dropped due to the normalization condition. Consequently, u belongs to the quotient space $\Lambda(\mathbb{S}^1)/\text{Möb}(\mathbb{S}^1)$ and the construction above provide a realization of $T_{[0]}\mathcal{T}(\mathbb{D})$ as the Möbius normalized Zygmund class on $\mathbb{S}^1$

$$T_{[0]}\mathcal{T}(\mathbb{D}) = \Lambda(\mathbb{S}^1)/\text{Möb}(\mathbb{S}^1).$$

The tangent space in the B model realization of universal Teichmüller space is obtained similarly by considering the infinitesimal solutions of the corresponding Beltrami equation. Now, the one-parameter family $f^{t\mu}$ is of the form

$$f^{t\delta\mu}(z) = z + t\delta z + O(t^2),$$

with the function δz being holomorphic on $\mathbb{D}^*$. It thus admits an expansion in $\mathbb{D}^*$ as

$$f^{t\delta\mu}(z) = z\Big(1 + \frac{tc_2}{z^2} + \frac{tc_3}{z^3} + \dots\Big). \tag{4.19}$$

The corresponding holomorphic quadratic differential, is then obtained via the (infinitesimal) Schwarzian derivative $\partial_w^3 f^{t\delta\mu}$ and can also be expanded in $\mathbb{D}^*$ as

$$h(z) = \frac{1}{z^4}\left(h_0 + \frac{h_1}{z} + \frac{h_2}{z^2} + \dots\right), \tag{4.20}$$

where the coefficients h_k are related to those in (4.19) via

$$h_{k-2} = c_k(k - k^3), \qquad k \geq 2. \tag{4.21}$$

Thus, the tangent space $T_{[0]}\mathcal{T}(\mathbb{D})$ is now realized as the space of holomorphic quadratic differentials on $\mathbb{D}^*$ with Laurent expansion (4.20).

The relation between the coefficients in the A and B models now becomes quite simple. For the c_k coefficients one gets

$$c_k = \frac{1}{\pi}\int_{\mathbb{D}} \delta\mu(z) z^{k-2} dx\, dy \tag{4.22}$$

and for the u_k

$$u_k = \frac{\mathrm{i}}{\pi}\int_{\mathbb{D}} \delta\bar{\mu}(z) \bar{z}^{k-2} dx\, dy. \tag{4.23}$$

Thus, in view of (4.21) we have

$$u_k = \mathrm{i}\frac{\bar{h}_{k-2}}{(k - k^3)}, \qquad k \geq 2, \tag{4.24}$$

see [65] for a proof.

The relation above could in fact be expected from the identification of $T_{[0]}\mathcal{T}(\mathbb{D})$ with the space of harmonic Beltrami coefficient HBD($\mathbb{D}$), in which the infinitesimal Beltrami coefficient $\delta\mu$ is given in terms of a dual holomorphic quadratic differential $q \in \text{HQD}(\mathbb{D})$. Writing

$$q(z) = \sum_{k\geq 0} q_k z^k \tag{4.25}$$

for the Laurent expansion of q, one can explicitly find δz by integration of

$$\partial_{\bar z}\delta z = -\frac{(1-|z|^2)^2}{2}\overline{q(z)}.$$

For $z \in \mathbb{D}$ we get

$$\delta z(z) = \frac{1}{2}\sum_{k\geq 2}\frac{\bar q_{k-2}\bar z^{k-1}}{(k-k^3)}\left[k(k+1) - 2(k^2-1)|z|^2 + k(k-1)|z|^4\right] + F(z),$$

where F is some holomorphic function on $\mathbb{D}$, and, by reflection symmetry $\delta z(z) = -z^2\overline{\delta z(1/\bar z)}$, for $z \in \mathbb{D}^*$

$$\delta z(z) = -\frac{1}{2}\sum_{k\geq 2}\frac{q_{k-2}\bar z^{-k-1}}{(k-k^3)}\left[k(k+1)|z|^4 - 2(k^2-1)|z|^2 + k(k-1)\right] - z^2\overline{F(1/\bar z)}.$$

We may then write F as

$$F(z) = \sum_{k\geq 0} v_k z^k$$

and restrict δz to $\mathbb{S}^1$ to get

$$\begin{aligned}\delta z(e^{\mathrm{i}\theta}) &= \sum_{k\geq 2}\frac{\bar q_{k-2}e^{-(k-1)\mathrm{i}\theta}}{(k-k^3)} + v_0 + v_1 e^{\mathrm{i}\theta} + v_2 e^{2\mathrm{i}\theta} + \sum_{k\geq 2} v_{k+1}e^{(k+1)\mathrm{i}\theta}\\ &= -\sum_{k\geq 2}\frac{q_{k-2}e^{(k+1)\mathrm{i}\theta}}{(k-k^3)} - \bar v_0 e^{2\mathrm{i}\theta} - \bar v_1 e^{\mathrm{i}\theta} - \bar v_2 - \sum_{k\geq 2}\bar v_{k+1}e^{-(k-1)\mathrm{i}\theta}.\end{aligned}$$

Thus we get $v_0 = -\bar v_2$, $v_1 = -\bar v_1$, $v_{k+1} = \mathrm{i}u_k$ for $k \geq 2$ and

$$\begin{aligned}\delta z(z) = \frac{1}{2}\sum_{k\geq 2}\frac{\bar q_{k-2}\bar z^{k-1}}{(k-k^3)}&\left[k(k+1) - 2(k^2-1)|z|^2 + k(k-1)|z|^4\right]\\ &+ v_0 + v_1 z + v_2 z^2 - \sum_{k\geq 2}\frac{q_{k-2}z^{k+1}}{(k-k^3)}, \quad z \in \mathbb{D}\end{aligned} \tag{4.26}$$

Note that, because of the normalization condition imposing δz to vanish at -1, $-\mathrm{i}$ and 1, the coefficients v_0, v_1, v_2 are completely determined by the q_{k-2}, $k \geq 2$. From now on we will drop these coefficients, understanding that they acquire the necessary values to make δz vanish at $-1, -\mathrm{i}, 1$.

We can now read off the Fourier coefficients of the Zygmund function u. With our choice of coefficients for q as above, u is simply given by

$$u(e^{\mathrm{i}\theta}) = \sum_{k\neq -1,0,1} u_k e^{\mathrm{i}k\theta},$$

with

$$u_k = \mathrm{i}\frac{q_{k-2}}{(k-k^3)}, \qquad u_{-k} = \bar u_k \tag{4.27}$$

In particular, the dual quadratic differential to $\delta\mu$, in the A model description, relates to the Bers embedding quadratic differential by the simple reflection rule

$$q(z) \in \mathrm{HQD}(\mathbb{D}) \longmapsto h(z) = \overline{q(1/\bar z)}\frac{1}{z^4} \in \mathrm{HQD}(\mathbb{D}^*). \tag{4.28}$$

For the later purposes, we now note that in all the discussions of the B-model above we could have replaced $\mathbb{D}$ and $\mathbb{D}^*$. In fact, this is the choice made in some of the references, see e.g. [37]. In this case one works with bounded Beltrami differentials on $\mathbb{D}^*$, solves the Beltrami equation continuing μ to vanish in $\mathbb{D}$ and gets a univalent holomorphic function on $\mathbb{D}$ whose Schwarzian derivative produces a holomorphic quadratic differential on the unit disc. The analogues of (4.19) and (4.20) in this realization of $\mathcal{T}(\mathbb{D})$ are then given by

$$\hat{f}^{t\mu}(z) = z + t\delta\hat{z} = z(1 + t\hat{c}_2 z^2 + t\hat{c}_3 z^3 + \ldots), \qquad z \in \mathbb{D}, \tag{4.29}$$

and

$$\hat{h}(z) = \hat{h}_0 + \hat{h}_1 z + \hat{h}_2 z^2 + \ldots \qquad z \in \mathbb{D}, \tag{4.30}$$

where we have denoted the quantities arising in this "$\hat{\text{B}}$-model" by letters with an extra hat. Note that there is an extra minus as compared to (4.21) in the relation between the coefficients in this realization

$$\hat{h}_{k-2} = -\hat{c}_k(k - k^3). \tag{4.31}$$

Also note, we can always map a holomorphic function inside the disc to an anti-holomorphic function outside by reflection $z \to 1/\bar{z}$. By applying this to the quadratic differential (4.30) we get a new anti-holomorphic quadratic differential $h(\bar{z})$ outside of the disc by complex conjugating

$$\overline{h(\bar{z})} = \hat{h}\left(\frac{1}{z}\right)\frac{1}{z^4} = \frac{1}{z^4}\left(\hat{h}_0 + \frac{\hat{h}_1}{z} + \frac{\hat{h}_2}{z^2} + \ldots\right) \qquad \text{in} \quad z \in \mathbb{D}^*. \tag{4.32}$$

This gives the same expression as in (4.20), but with the change $z \to \bar{z}$. We shall also make use, in what follows, of this $\hat{\text{B}}$ model of $\mathcal{T}(\mathbb{D})$ in terms of univalent anti-holomorphic functions on $\mathbb{D}^*$ or their associated anti-holomorphic quadratic differentials.

We now need the relation between the $\hat{c}$-coefficients in the realization of the $\hat{\text{B}}$-model in terms of Beltrami coefficients on $\mathbb{D}^*$ and the Fourier coefficients u_k of the Zygmund functions u of the A-model. The derivation is a straightforward adaptation of the proof in [65]. One finds

$$\hat{c}_k = -\frac{1}{\pi}\int_{\mathbb{D}^*} \frac{\mu(z)}{z^{k+2}} dx\, dy. \tag{4.33}$$

In relating it to the A-model Fourier coefficients we use the fact that the A-model Beltrami coefficient on $\mathbb{D}^*$ is obtained by reflection. Thus, the integral in (4.33) can be taken over the unit disc with the reflected Beltrami given by

$$\mu\left(\frac{1}{\bar{z}}\right) = \overline{\mu(z)}\frac{z^2}{\bar{z}^2}.$$

Substituting this to the integral, and taking into account the change of integration measure $dx\, dy \to -dx\, dy/|z|^4$ we get

$$\hat{c}_k = \frac{1}{\pi}\int_{\mathbb{D}} \overline{\mu(z)}\bar{z}^{k-2} dx\, dy.$$

The c and $\hat{c}$-coefficients are thus related by complex conjugation and the A and $\hat{\mathrm{B}}$ model coefficients by

$$u_k = \mathrm{i}\bar{c}_k = \mathrm{i}\hat{c}_k. \tag{4.34}$$

We finish the presentation on the infinitesimal description of universal Teichmüller space with some useful identities following the identification of $T_{[0]}\mathcal{T}(\mathbb{D})$ with harmonic Beltrami coefficients. These identities relate the variations of quasiconformal maps δz and its derivatives and are related to the holomorphicity of the dual quadratic differential of $\delta\mu$. We shall need them in chapter 5, for the infinitesimal description of the phase space of 2+1 AdS gravity, and chapter 6, for our considerations of the symplectic structure on that phase space.

First, we have the identity

$$\partial_z \delta\mu + \frac{2\bar{z}}{(1-|z|^2)}\delta\mu, \tag{4.35}$$

which is a direct consequence of the holomorphicity of the dual quadratic differential

$$q = \frac{-2\delta\bar{\mu}}{(1-|z|^2)^2}.$$

The second identity is more non-trivial, see [49],

$$2\frac{\bar{z}\delta z + z\delta\bar{z}}{(1-|z|^2)} + \partial_z\delta z + \partial_{\bar{z}}\delta\bar{z} = 0. \tag{4.36}$$

It can be directly verified from the mode expansion (4.26) of δz obtained previously by direct integration of the infinitesimal Beltrami equation. It is easy to obtain a geometric interpretation of this condition as an area preserving condition for the map $f_{t\delta\mu}$. In fact, the hyperbolic area form in $\mathbb{D}$ is given by

$$da_I = \frac{4dz \wedge d\bar{z}}{(1-|z|^2)^2}$$

and its variation is easily shown to vanish as a consequence of (4.36)

$$\delta da_I = \Big(2\frac{\bar{z}\delta z + z\delta\bar{z}}{(1-|z|^2)} + \partial_z\delta z + \partial_{\bar{z}}\delta\bar{z}\Big)da_I = 0.$$

4.2 Maximal surfaces in AdS_3

4.2.1 Existence and uniqueness

We now describe the relation between universal Teichmüller theory and maximal surfaces in AdS_3, established in [39]. The general idea comes from the possibility of associating to any homeomorphism of $\mathbb{S}^1$ a closed acausal curve on the conformal boundary $\partial_\infty\mathrm{AdS}_3$ coming from the (2-to-1) identification between $\partial_\infty\mathrm{AdS}_3$ and $\mathbb{S}^1 \times \mathbb{S}^1$. In fact, we have given in section 2.2 the converse construction. There, we have defined, given a spacelike surface S in AdS_3, a pair of homeomorphisms $\phi_\pm = \Phi^S_\pm\big|_{\partial_\infty S} : \partial_\infty S \to \mathbb{S}^1$ which may then be composed into a

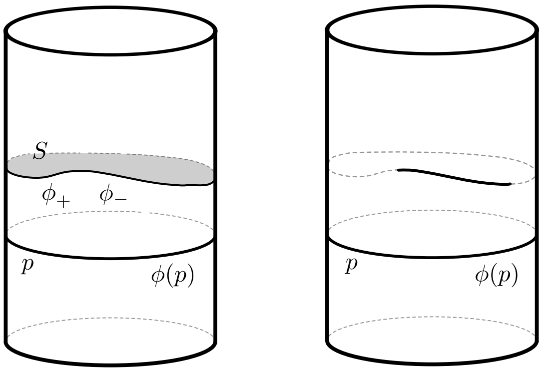

Figure 4.1: Left: Circle homeomorphisms form acausal curves. Right: Acausal curves as graphs of homeomorphisms the circle.

homeomorphism $\phi = \phi_- \circ \phi_+^{-1} : \mathbb{S}^1 \to \mathbb{S}^1$. The homeomorphisms $\phi_\pm$ were obtained by, first, fixing an arbitrary geodesic spacelike plane in AdS_3, which maybe identified with the unit disc and looking at the intersection of its conformal boundary with null geodesics, in each family $\mathcal{L}_\pm$, starting at points in the conformal boundary of S, see figure 4.1. Thus, starting from a homeomorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$, we now construct the corresponding acausal curve by taking the set of intersection points between the left-moving null geodesic starting at points in $\mathbb{S}^1$ and the right-moving null geodesic starting at the corresponding image by ϕ. Here the invertibility of ϕ is sufficient for this set to be an acausal curve.

This construction, in particular, identifies $\mathcal{T}(\mathbb{D})$ with a special class of closed acausal curves in the conformal boundary of AdS_3 and we would like to understand the geometrical meaning of that class. We will then have to look at spacelike surfaces in AdS_3 intersecting the conformal boundary $\partial_\infty \mathrm{AdS}_3$ along graphs of quasisymmetric homeomorphisms of $\mathbb{S}^1$. As shown in [39] any closed acausal curves in $\partial_\infty \mathrm{AdS}_3$ is the conformal boundary of a maximal spacelike surface and it will be possible to characterize the quasisymmetry condition in terms of the extrinsic geometry of such surfaces.

We do not wish here to present a complete proof for the existence and uniqueness of maximal surfaces given quasisymmetric boundary conditions. This can be found in [39]. Our aim is to give only a brief overview of this result for the convenience of the reader.

We start with the existence of maximal surfaces with given boundary curve at infinity. This relies on an analogous existence result for compact spacelike maximal surfaces with boundaries laying on constant radius cylinders in AdS_3 and on convexity and causality arguments, ensuring the convergence of a sequence of such compact spacelike maximal surfaces to a maximal spacelike surface with the prescribed asymptotic boundary. Thus, thinking of AdS_3 as a product $\mathbb{R} \times \mathbb{D}$ one describe spacelike surfaces as graphs of smooth functions $F : \mathbb{D} \to \mathbb{R}$ satisfying

$$|\partial_z F|^2 < \frac{1}{(1+|z|^2)^2}. \tag{4.37}$$

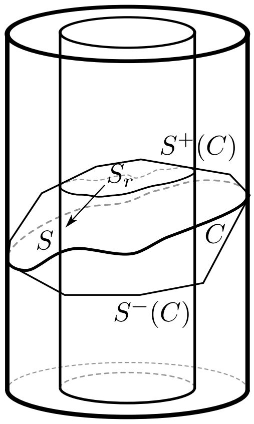

Figure 4.2: Convergent sequence of compact maximal surfaces with boundaries in the future boundary component of the convex core of an acausal curve.

Thus only says its unit normal vector is timelike. The maximality condition then becomes a second order differential equation for F

$$\partial_z\partial_{\bar{z}}F + \frac{2z}{(1-|z|^4)}\partial_z F + \frac{2\bar{z}}{(1-|z|^4)}\partial_{\bar{z}}F = 0, \tag{4.38}$$

which states the vanishing of the mean curvature. These expressions are easily obtained using global cylindrical coordinates (1.2) with $z = re^{\mathrm{i}\theta}$.

Then, given a closed acausal curve C in $\partial_\infty \mathrm{AdS}_3$, one consider its convex hull $K(C)$, the smallest convex domain in AdS_3 containing C. Its boundary is formed by two connected components $S^\pm(C)$, called the future and past boundaries, having C as their asymptotic boundary. For $0 < r < 1$, it is shown that there exists a compact maximal surface S_r, contained in $K(C)$, whose boundary is the intersection of $S^+(C)$ with the radius r cylinder in AdS_3. Further, S_r can be viewed as the graph of some function F_r on disc $\mathbb{D}_r$ of radius r. It is then possible to construct a convergent sequence F_{r_k} such that the limiting function $F : \mathbb{D} \to \mathbb{R}$ still satisfies equation (4.38) with the given boundary conditions thus describing a maximal spacelike surface with the prescribed asymptotic boundary, see figure 4.2.

The uniqueness result is only obtained assuming further regularity to the data at infinity. It is here that the quasisymmetry condition becomes relevant. Thus, [39] introduce the notion of the width $w(C)$ of the convex hull. It is defined as the supremum of the time distance between points in its future and past boundaries $S^\pm(C)$ and gives a partial measure of the extrinsic properties of the maximal surfaces intersecting $\partial_\infty \mathrm{AdS}_3$ along C. By causality/convexity reasons, the width cannot be greater than $\pi/2$, being strictly less than $\pi/2$ if and only if C is the graph of a quasisymmetric homeomorphism. On the other hand, the $w(C) < \pi/2$ condition is sufficient for the uniqueness result. First, it is shown to imply that the corresponding maximal surface has sectional curvature bounded from above by a negative constant. Then, based on convexity properties, this maximal surface with uniformly negative sectional curvature is shown to be unique among complete maximal graphs with the given asymptotic boundary and bounded second fundamental form. This then gives a one-to-one correspondence between points in $\mathcal{T}(\mathbb{D})$ and maximal spacelike surfaces with bounded second fundamental form and uniformly negative sectional curvature.

Theorem 4.2.1. *(Bonsante and Schlenker [39]) Let $C \subset \partial_\infty \mathrm{AdS}_3$ be the graph of a quasi-symmetric homeomorphism of $\mathbb{S}^1$. Then there exist a unique maximal spacelike surface S in AdS_3 with asymptotic boundary $\partial_\infty S = C$, bounded second fundamental form and uniformly negative sectional curvature.*

4.2.2 Minimal Lagrangian and harmonic diffeomorphisms

Note that the maximal surfaces come equipped with their induced metric I and extrinsic curvature $I\!I$ coming from the ambient AdS_3 geometry. Similarly to the compact case, see section 2.2, this initial data $(I, I\!I)$ on maximal surfaces can be described in terms of quasiconformal minimal Lagrangian diffeomorphisms of the unit disc. We remind the reader, minimal Lagrangian diffeomorphisms are area preserving diffeomorphisms whose graphs are minimal in the product $\mathbb{D} \times \mathbb{D}$ with the product metric. The existence and uniqueness result just described is then equivalent to the existence and uniqueness of quasiconformal minimal Lagrangian extensions of quasisymmetric homeomorphisms of $\mathbb{S}^1$ to the interior of the disc. This is an interesting result in its own right from the point of view of Teichmüller theory. It is closely related to Schoen's conjecture [68] on the existence and uniqueness of harmonic extensions. This is to say, the interplay between AdS geometry and Teichmüller theory, far from being only a mathematical curiosity, proves to be an interesting tool for the development of new results in Teichmüller theory.

Concretely, harmonic diffeomorphisms $\Phi : \mathbb{D} \to \mathbb{D}$ are critical points of the energy functional

$$\mathcal{E}(\Phi) = \frac{1}{2\mathrm{i}} \int_{\mathbb{D}} \frac{4dz \wedge d\bar{z}}{(1-|z|^2)^2} \Big(|\partial\Phi|^2 + |\bar{\partial}\Phi|^2 \Big), \tag{4.39}$$

where

$$|\partial\Phi| = \frac{1-|z|^2}{1-|\Phi|^2} |\partial_z \Phi| \qquad |\bar{\partial}\Phi| = \frac{1-|z|^2}{1-|\Phi|^2} |\partial_{\bar{z}} \Phi| \tag{4.40}$$

are the holomorphic and anti-holomorphic energy densities of Φ. The corresponding Euler-Lagrange equation is given by

$$\partial_z \partial_{\bar{z}} \Phi + \frac{2\bar{\Phi}}{1-|\Phi(z)|^2} \partial_z \Phi \partial_{\bar{z}} \Phi = 0. \tag{4.41}$$

One can then show that the above equation for the harmonicity of Φ is equivalent to holomorphicity of an associated quadratic differential, the Hopf differential of Φ,

$$\mathrm{Hopf}(\Phi) = \frac{4\partial_z \Phi \partial_z \bar{\Phi}}{(1-|\Phi(z)|^2)^2} dz^2. \tag{4.42}$$

This can be checked by a direct computation.

The relation to minimal Lagrangian diffeomorphisms is then obtained from the following lemma, due to Schoen [68], see also [40].

Lemma 4.2.1.1. *A diffeomorphism $f : \mathbb{D} \to \mathbb{D}$ is minimal Lagrangian if and only if f may be decomposed as $f = \Phi_- \circ \Phi_+^{-1}$ in terms of two harmonic maps $\Phi_\pm : \mathbb{D} \to \mathbb{D}$ whose Hopf differentials add up to zero*

$$\mathrm{Hopf}(\Phi_+) + \mathrm{Hopf}(\Phi_-) = 0.$$

As we shall see in the next section, this relationship leads to a particularly simple expressions for the geometric data on the maximal surface. This decomposition is in fact very natural from the maximal surface's point of view with the harmonic maps $\Phi_\pm$ being interpreted as generalized Gauss maps associated to the maximal surface, see [40].

4.3 The Universal Phase Space

4.3.1 The expected generalization

We are now ready to describe the construction of the phase space of 2+1 AdS spacetimes with non-compact spatial topology. As seen in chapter 2 the phase space of spatially compact 2+1 AdS gravity is given by two copies of Teichmüller space of a initial Cauchy surface or, equivalently, by the cotangent bundle over a single copy of that space. It thus seem natural to expect that the generalization for the phase space of spatially non-compact 2+1 AdS spacetimes, say, with topology $\mathbb{R} \times \mathbb{D}$, would be given by two copies of universal Teichmüller space or, else, by its cotangent bundle. On the other hand, it seems from the discussion of the previous section, that a single quasisymmetric homeomorphism of the circle (single point in $\mathcal{T}(\mathbb{D})$) is sufficient to specify all the data (I, II) for an "initial value" description. We are thus left with the question whether a second phase space direction, that is, a second copy of $\mathcal{T}(\mathbb{D})$, is indeed necessary.

The case for a second direction on the phase space is related to the asymptotic, non-geometric, degrees of freedom introduced in the presence of a conformal boundary, as discussed in the previous chapter, section 3.2. We remind the reader that the equivalence relation between AdS spacetimes considered here is not given by the action of full group of diffeomorphisms, but only that of the group of asymptotically trivial diffeomorphisms, see 3.2. It is clear, form the constructions of [39], that the initial data (I, II) determined by a single quasisymmetric homeomorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ is insensitive to any spacetime diffeomorphism preserving $\mathrm{graph}(\phi) \subset \partial_\infty \mathrm{AdS}_3$, here thought of as a closed acausal curve on the conformal boundary. In particular, from the point of view of 2+1 gravity, applying an asymptotically non-trivial diffeomorphism which preserves the maximal surface, in other words, a purely spatial diffeomorphism, produces an new spacetime inequivalent to the one we started from.

We shall argue, in this section, that such purely spatial diffeomorphism is nothing but a quasiconformal deformation of the maximal surface and, therefore, is parametrized by another point in $\mathcal{T}(\mathbb{D})$. Then, our parametrization of the phase space of 2+1 AdS gravity by $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ can be interpreted as the quasiconformal deformation space of domains of dependence of a geodesic surface in AdS_3. The first phase space direction is given by deformations of its boundary curve. It is thus interpreted as a "geometric" direction since it completely determines the initial data $(I, I\!I)$ on a new maximal surface. The second direction is then given by spatial quasiconformal deformations of this new maximal surface itself. It is interpreted as "non-geometric" since it does not alter the maximal surface's initial data.

Note that the spacetimes we consider here are not globally hyperbolic and, therefore, the initial data on a spacelike surface is not enough to characterize the whole spacetime geometry. Our parametrization will then only make sense if a well defined analytic continuation of the spacetime metric on the maximal surface's domain of dependence can be performed to a region beyond the Cauchy horizon. This will be given in chapter 5, where we shall obtain an identification between the generators of quasiconformal and asymptotic deformations. Such an identification not only provides a well defined analytic continuation of the initial data description beyond the the Cauchy horizon, it also proves that both our deformation directions are asymptotically non-trivial in the sense of chapter 3. Even further, this identification also shows that the quasiconformal deformations described here contain all Brown-Henneaux excitations of the given reference spacetime, thus justifying our characterization of the quasiconformal deformation space $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ as the phase space of 2+1 AdS gravity on $\mathbb{R} \times \mathbb{D}$.

We would also like to remark that our focus on spacetimes with topology $\mathbb{R} \times \mathbb{D}$ is not at all restrictive. We may later obtain nontrivial topologies by taking quotients by appropriate pairs of Fuchsian groups. Similarly to the two dimensional case, the phase space space of fixed spatial topology AdS manifolds will then embed in $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ in the same manner fixed topology Teichmüller space embeds in $\mathcal{T}(\mathbb{D})$. They can then be recovered by restricting our construction to pairs of quasisymmetric homeomorphisms invariant under appropriate discrete subgroups of $PSU(1,1)$. In this sense, we shall call the phase space $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ the "universal phase space" of AdS 2+1 gravity.

4.3.2 The generalized parametrization

We start with the construction of the initial data on a maximal surface in an AdS spacetime from a pair of points $[\mu_+], [\mu_-] \in \mathcal{T}(\mathbb{D})$. We shall here consider the real analytic realization of $\mathcal{T}(\mathbb{D})$ as the space of normalized quasisymmetric homeomorphisms of the unit circle. Let's denote the homeomorphisms associated to $[\mu_\pm]$ by $\phi_\pm : \mathbb{S}^1 \to \mathbb{S}^1$. As will become clear below, it will be useful to extend $\phi_\pm$ to two quasiconformal maps $f_{\mu_\pm}$ from a "base point" reference unit

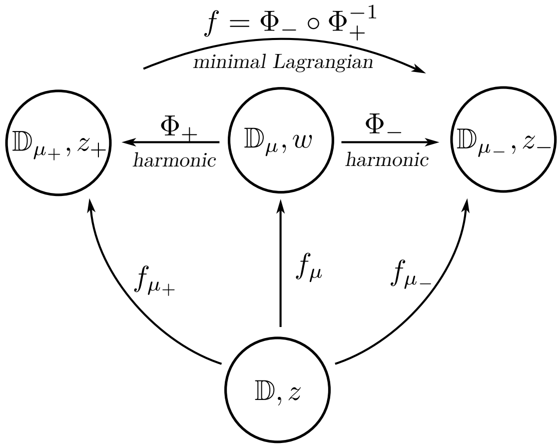

Figure 4.3: Diagram of quasiconformal deformations.

disc, representing the preferred point in $\mathcal{T}(\mathbb{D})$. We call the complex coordinate in this reference disc z, see Figure 4.3, and interpret the quasiconformal deformations $f_{\mu_\pm}$ as defining complex coordinates $z_\pm = f_{\mu_\pm}(z)$ on the target discs $\mathbb{D}_{\mu_\pm} = f_{\mu_\pm}(\mathbb{D})$.

Each of these discs then has its standard hyperbolic metric

$$I_\pm = \frac{4|dz_\pm|^2}{(1-|z_\pm|^2)^2}$$

which can then be used to represent the given points in $\mathcal{T}(\mathbb{D})$. Note that $I_\pm$ are to be considered as representatives of inequivalent classes of bounded Beltrami coefficients. Although $I_\pm$ are isometric, the isometry mapping I_+ to I_- acts nontrivially at infinity changing corresponding the boundary homeomorphisms. The existence of a preferred point in $\mathcal{T}(\mathbb{D})$ thus becomes quite helpful in avoiding confusion. In fact, when pulled back to the reference disc, the metrics $I_\pm$ explicitly involve the Beltrami coefficients they are to represent

$$I_\pm = \frac{4|\partial_z f_{\mu_\pm}|^2}{(1-|f_{\mu_\pm}(z)|^2)^2}|dz + \mu_\pm d\bar{z}|^2. \tag{4.43}$$

It is in this sense only that we shall use the hyperbolic metrics $I_\pm$ on $\mathbb{D}$ as representatives of points in $\mathcal{T}(\mathbb{D})$.

This discussion can, perhaps, be made more clear in the context of nontrivial spatial topology. Thus, let's fix $(S, X) = \mathbb{D}/\Gamma$ a Riemann surface (compact or not) and consider the corresponding Teichmüller space $\mathcal{T}(S)$. We remind the reader, our description of Teichmüller space as the space of quasiconformal deformations of (S, X) associates to each Beltrami coefficient μ a deformation of the reference Fuchsian group Γ into $\Gamma_\mu = f_\mu \circ \Gamma \circ f_\mu^{-1}$ and defines a new Riemann surface $(S, X_\mu) = \mathbb{D}_\mu/\Gamma_\mu$. Now, it becomes clear that the hyperbolic metrics in (S, X) and (S, X_μ) cannot be globally isometric since, for example, they assign, for corresponding nontrivial cycles, different values of length (associated with the traces of the corresponding generators A and A_μ).

Note that there are many equivalent quasiconformal maps $f_{\mu_\pm}$ in the sense of universal Teichmüller theory, i.e. having the same restrictions $\phi_\pm$ to the unit circle. Therefore, a word is in order about which quasiconformal extensions $f_{\mu_\pm}$ of quasisymmetric boundary homeomorphisms $\phi_\pm$ are to be considered here. We shall see below that, for our purposes, the composition $f_{\mu_-} \circ f_{\mu_+}^{-1}$ will be imposed to satisfy certain property, making it unique given the boundary values. Apart from this restriction, the extensions $f_{\mu_\pm}$ are arbitrary and, for our construction of the phase space, it will not matter which specific extension is chosen.

Let us now consider the composition $\phi = \phi_- \circ \phi_+^{-1} : \mathbb{S}^1 \to \mathbb{S}^1$. This is also a quasisymmetric homeomorphism and, according to [39], there is a unique maximal surface S (with bounded second fundamental form and uniformly negative sectional curvature) in AdS_3 whose asymptotic boundary is the graph of ϕ. Then, let $f : \mathbb{D}_+ \to \mathbb{D}_-$ denote the minimal Lagrangian extension of ϕ, see section 4.2. We fix the arbitrariness in $f_{\mu_\pm}$ (to some extent) by requiring their composition to agree with the map f

$$f_{\mu_-} \circ f_{\mu_+}^{-1} = f. \tag{4.44}$$

As in the compact case, see lemma 2.2.1.1, the knowledge of f is sufficient to reconstruct both the first and second fundamental forms on the maximal surface up to diffeomorphisms. These are again constructed from an associated operator $b : T\mathbb{D}_+ \to T\mathbb{D}_+$ satisfying

1. $\det b = 1$;
2. b is self-adjoint with respect to I_+;
3. $d^{D^+} b = 0$, where D^+ is the Levi-Civita connection of I_+;
4. $f^* I_- = I_+(b\,\cdot\,, b\,\cdot\,)$

via

$$I = \frac{1}{4} I_+(E + b\,\cdot\,, E + b\,\cdot\,), \qquad I\!I = -IJ(E+b)^{-1}(E-b). \tag{4.45}$$

As in section 2.2, E is the identity operator and J is the almost-complex structure induced by I. Conditions (1-4) on b are again equivalent to the Gauss-Codazzi equations (2.21) for the pair $(I, I\!I)$ and we may construct the spacetime metric in equidistant coordinates

$$g = -d\tau^2 + \cos^2\tau I + 2\sin\tau\cos\tau I\!I + \sin^2\tau I\!I I^{-1} I\!I. \tag{4.46}$$

Note that this construction only makes use of the "difference" $\phi = \phi_- \circ \phi_+^{-1}$ between the quasisymmetric homeomorphisms. This can thus be interpreted as a "geometric" direction on the phase space. The other direction is encoded in the particular way ϕ decomposes into $\phi_\pm$, giving rise to a quasiconformal deformation describing the relation between the maximal surface conformal structure and the preferred reference disc. Again, it becomes convenient to pull-back the maximal surface data to the reference disc to obtain expressions with an explicit dependence on $([\mu_+], [\mu_-])$.

We then need a more explicit description of the operator $b : T\mathbb{D}_+ \to T\mathbb{D}_+$ associated with the minimal Lagrangian diffeomorphism f. In complex coordinates z_+ on the source disc of f it is possible to use conditions 1,2 and 4 above to write b as

$$b = |\partial f| \left(\partial_{z_+} dz_+ + \lambda \partial_{z_+} d\bar{z}_+ + \bar{\lambda} \partial_{\bar{z}_+} dz_+ + \partial_{\bar{z}_+} d\bar{z}_+ \right) , \tag{4.47}$$

where $\lambda = \partial_{\bar{z}_+} f / \partial_{z_+} f$ is the Beltrami coefficient of f and

$$|\partial f| = \frac{1 - |z_+|^2}{1 - |f|^2} |\partial_{z_+} f|$$

its holomorphic energy density. It is then easy to compute the induced metric of the maximal surface via (4.45)

$$I = \frac{|\partial f|(|\partial f| + 1)}{(1 - |z_+|^2)^2} \left(2|dz_+|^2 + \bar{\lambda} dz_+^2 + \lambda d\bar{z}_+^2 \right) , \tag{4.48}$$

the almost-complex structure of I

$$J = \mathrm{i}|\partial f| \left(\partial_{z_+} dz_+ + \lambda \partial_{z_+} d\bar{z}_+ - \bar{\lambda} \partial_{\bar{z}_+} dz_+ - \partial_{\bar{z}_+} d\bar{z}_+ \right)$$

and the operator $(E + b)^{-1}(E - b)$

$$(E + b)^{-1}(E - b) = -\frac{|\partial f|}{|\partial f| + 1} \left(\lambda \partial_{z_+} d\bar{z}_+ + \bar{\lambda} \partial_{\bar{z}_+} dz_+ \right) .$$

This leads, via (4.45), to the following expression for the second fundamental form

$$I\!I = -\mathrm{i} \frac{|\partial f|}{(1 - |z_+|^2)^2} (\bar{\lambda} dz_+^2 - \lambda d\bar{z}_+^2). \tag{4.49}$$

We may now pull-back $(I, I\!I)$ to the reference z-disc with the map f_{μ_+}. The transformation of the Beltrami coefficient λ and the holomorphic energy density $|\partial f|$ are easily obtained computing derivatives of $f_{\mu_-} = f \circ f_{\mu_+}$

$$\lambda \circ f_{\mu_+} \frac{\partial_{\bar{z}} \bar{f}_{\mu_+}}{\partial_z f_{\mu_+}} = \frac{\mu_- - \mu_+}{1 - \mu_- \bar{\mu}_+}, \qquad |\partial f| \circ f_{\mu_+} |\partial_z f_{\mu_+}| = |\partial f_{\mu_-}| \frac{|1 - \bar{\mu}_+ \mu_-|}{1 - |\mu_+|^2}.$$

Using the area preserving condition for f, we then get the following expressions for the maximal surface data

$$\begin{aligned} I = \frac{|\partial f_{\mu_+}||\partial f_{\mu_-}|}{(1 - |z|^2)^2} & \left(1 + \frac{|1 - \mu_- \bar{\mu}_+|}{(1 - |\mu_-|^2)^{1/2}(1 - |\mu_+|^2)^{1/2}} \right) \left(2 \frac{1 - |\mu_-|^2 |\mu_+|^2}{|1 - \mu_- \bar{\mu}_+|} |dz|^2 \right. \\ & \left. + \frac{\bar{\mu}_+(1 - |\mu_-|^2) + \bar{\mu}_-(1 - |\mu_+|^2)}{|1 - \mu_- \bar{\mu}_+|} dz^2 + \frac{\mu_+(1 - |\mu_-|^2) + \mu_-(1 - |\mu_+|^2)}{|1 - \mu_- \bar{\mu}_+|} d\bar{z}^2 \right) \end{aligned}$$

$$\begin{aligned} I\!I = \mathrm{i} \frac{|\partial f_{\mu_+}||\partial f_{\mu_-}|}{(1 - |z|^2)^2} & \left(2 \frac{\mu_+ \bar{\mu}_- - \bar{\mu}_+ \mu_-}{|1 - \mu_- \bar{\mu}_+|} |dz|^2 + \frac{\bar{\mu}_+(1 + |\mu_-|^2) - \bar{\mu}_-(1 + |\mu_+|^2)}{|1 - \mu_- \bar{\mu}_+|} dz^2 \right. \\ & \left. - \frac{\mu_+(1 + |\mu_-|^2) - \mu_-(1 + |\mu_+|^2)}{|1 - \mu_- \bar{\mu}_+|} d\bar{z}^2 \right) . \end{aligned} \tag{4.50}$$

Again,

$$|\partial f_{\mu_\pm}| = \frac{1-|z|^2}{1-|f_{\mu_\pm}|^2}|\partial_z f_{\mu_\pm}|,$$

are the holomorphic energy densities of $f_{\mu_\pm}$.

Let's now give further explanation on the ambiguity that entered into the above construction. Recall, the only condition imposed on the quasiconformal extensions $f_{\mu_\pm}$ of the quasisymmetric maps $\phi_\pm$ was that their composition $f_{\mu_-}\circ f_{\mu_+}^{-1}$ was given by the minimal Lagrangian diffeomorphism extending $\phi_-\circ\phi_+^{-1}$. One can now see that nothing depends on the remaining extension ambiguity. Indeed, choosing different extensions for $\phi_\pm$, say $\tilde f_{\mu_\pm}$, which are in the same universal Teichmüller class as $f_{\mu_\pm}$ and still satisfy (4.44), that is, $\tilde f_{\mu_-}\circ\tilde f_{\mu_+}^{-1} = f$, we obtain another pair $(\tilde I, \tilde{I\!I})$, as well as the corresponding spacetime metric

$$\tilde g = -d\tau^2 + \cos^2\tau\tilde I + 2\sin\tau\cos\tau\tilde{I\!I} + \sin^2\tau\tilde{I\!I}\tilde I^{-1}\tilde{I\!I}.$$

It is, however, clear that this metric can be mapped into (4.46) by the (purely spatial) diffeomorphism $\tilde f_{\mu_+}\circ f_{\mu_+}^{-1}$. This, in its turn, is easily seen to be asymptotically trivial in the sense of Teichmüller theory since its restriction to the boundary is nothing but the identity homeomorphism

$$\tilde f_{\mu_+}\circ f_{\mu_+}^{-1}\Big|_{\mathbb{S}^1} = \phi_+\circ\phi_+^{-1} = \mathrm{Id}.$$

The spacetime metrics g and $\tilde g$ should, therefore, be considered equivalent and nothing in the above construction depends on which particular extension of $\phi_\pm$ are chosen, provided the minimal Lagrangian condition (4.44) holds.

4.3.3 The Mess map

We would now like to describe a generalization of the map (2.36), following from a cotangent bundle parametrization of the universal phase space. One direction of this map, namely $T^*\mathcal{T}(\mathbb{D})\to\mathcal{T}(\mathbb{D})\times\mathcal{T}(\mathbb{D})$, arises in the same way as in the compact setting. The only non-trivial point is the existence and uniqueness of solutions of the Gauss equation (2.29) on $\mathbb{D}$. This can be found in [41] to which we refer the reader for more details. Note that, although the treatment in this reference is carried out for CMC surfaces in the Minkowski space $\mathbb{R}^{2,1}$, it needs very little adaptation to the present situation. The converse direction is achieved using the harmonic decomposition of minimal Lagrangian diffeomorphism discussed in section 4.2.

Thus, given a point $([\mu], q)\in T^*\mathcal{T}(\mathbb{D})$ there exists a unique solution φ for the Gauss equation

$$4\partial_w\partial_{\bar w}\varphi = e^{2\varphi} - e^{-2\varphi}|q|^2 \tag{4.51}$$

making $I = e^{2\varphi}|dw|^2$ a complete Riemannian metric on $\mathbb{D}$. One also defines a symmetric bilinear form as the real part of the quadratic differential qdw^2

$$I\!I = \frac{1}{2}(qdw^2 + \bar q d\bar w^2).$$

Since $I\!I$ is traceless with respect to I, the holomorphicity of q is then equivalent to Codazzi equation and, by the fundamental theorem of AdS geometry 2.2.1, the pair $(I, I\!I)$ defines a maximal surface in an AdS spacetime. Here the interpretation in terms of deformations of the domain of dependence of a geodesic surface in AdS_3 is also valid, with the "geometric" and "non-geometric" directions given respectively by the vertical and horizontal directions of $T^*\mathcal{T}(\mathbb{D})$ and the reference domain of dependence determined by the pair

$$I_0 = \frac{4|dz|^2}{(1-|z|^2)^2}, \qquad I\!I_0 = 0$$

where $z = f_\mu^{-1}(w)$.

The converse construction is also easily obtained. Given a maximal surface in an AdS spacetime we may write its fundamental forms as

$$I = e^{2\varphi}, \qquad I\!I = \frac{1}{2}(qdw^2 + \bar{q}d\bar{w}^2)$$

where w is the isothermal coordinate of I, q is holomorphic with respect to w by the Codazzi equation and φ is obtained via the Gauss equation, thus determining a holomorphic quadratic differential qdw^2. Further, by comparing isothermal coordinate in our maximal surface with that of the fixed reference geodesic surface in AdS_3, we obtain a quasiconformal map $z \mapsto w = f_\mu(z)$, which determines the base point $[\mu]$ in $T^*\mathcal{T}(\mathbb{D})$.

To obtain the corresponding pair of points in $\mathcal{T}(\mathbb{D})$, we first construct associated hyperbolic metrics via formula (2.33)

$$I_\pm = e^{2\varphi}|dw \pm \mathrm{i}e^{-2\varphi}\bar{q}d\bar{w}|^2. \tag{4.52}$$

To see these are indeed hyperbolic, let

$$\theta^{z_\pm} = e^{\varphi}dw \pm \mathrm{i}e^{-\varphi}\bar{q}d\bar{w}, \qquad \theta^{\bar{z}_\pm} = e^{\varphi}d\bar{w} \mp \mathrm{i}e^{-\varphi}qdw$$

denote co-frame fields for the metrics $I_\pm$. Using the holomorphicity of q, it is easy to compute the spin connection associated with these fields

$$\omega^{z_\pm}{}_{z_\pm} = \partial_w\varphi dw - \partial_{\bar{w}}\varphi d\bar{w}, \qquad \omega^{\bar{z}_\pm}{}_{\bar{z}_\pm} = \partial_{\bar{w}}\varphi d\bar{w} - \partial_w\varphi dw$$

and, therefore, the associated curvature 2-form

$$R^{z_\pm}{}_{z_\pm} = -2\partial_w\partial_{\bar{w}}\varphi dw \wedge d\bar{w}, \qquad R^{\bar{z}_\pm}{}_{\bar{z}_\pm} = 2\partial_w\partial_{\bar{w}}\varphi dw \wedge d\bar{w}.$$

The scalar curvature is then given by

$$R = R^a{}_{bij}\theta^i_a\eta^{jk}\theta^b_k = -\frac{8\partial_w\partial_{\bar{w}}\varphi}{(e^{2\varphi} - e^{-2\varphi}|q|^2)} = -2$$

from Gauss equation (4.51).

We now obtain the point in $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ by reading from the hyperbolic metric $I_\pm$ the associated Beltrami coefficients. Note that the base point $[\mu]$, with respect to which the quadratic differential q is holomorphic, is not the preferred point in $\mathcal{T}(\mathbb{D})$. Thus we still have to consider another quasiconformal deformation f_μ, mapping the reference disc coordinate z to the maximal surface's coordinate $w = f_\mu(z)$. Writing

$$\nu_\pm = \pm \mathrm{i} e^{-2\varphi} \bar{q}$$

for the Beltrami associated to $I_\pm$ on the maximal surface, it is now just a matter of using the group structure (4.8) of $\mathcal{T}(\mathbb{D})$ to get the Beltrami coefficients on the reference disc

$$\mu_\pm = \frac{\mu \pm \nu \circ f_\mu (\partial_{\bar{z}} \bar{f}_\mu / \partial_z f_\mu)}{1 \pm \bar{\mu} \nu \circ f_\mu (\partial_{\bar{z}} \bar{f}_\mu / \partial_z f_\mu)}.$$

This gives an explicit description of the generalized Mess map in terms of Beltrami representatives of classes in $\mathcal{T}(\mathbb{D})$.

Let's now describe the inverse of this map. Thus, given $([\mu_+], [\mu_-])$, we again consider their quasisymmetric realizations $\phi_\pm : \mathbb{S}^1 \to \mathbb{S}^1$. Then, we take the harmonic decomposition, given by lemma (4.2.1.1), of the associated minimal Lagrangian extension $f : \mathbb{D} \to \mathbb{D}$ of the composition $\phi_- \circ \phi_+^{-1}$

$$f = \Phi_- \circ \Phi_+^{-1}, \qquad \mathrm{Hopf}(\Phi_-) = -\mathrm{Hopf}(\Phi_+).$$

As described above, the harmonicity (4.41) of $\Phi_\pm$ is equivalent to the holomorphicity of the Hopf differentials

$$\mathrm{Hopf}(\Phi_\pm) = \frac{4 \partial_w \Phi_\pm \partial_w \bar{\Phi}_\pm}{(1 - |\Phi_\pm|^2)^2} dw^2.$$

Note that, here, $w = \Phi_\pm^{-1}(z_\pm)$ denotes the coordinate on the source disc of $\Phi_\pm$ and needs not be the same as the reference disc coordinate z. Using the associated Beltrami differentials

$$\nu_\pm = \partial_{\bar{w}} \Phi_\pm / \partial_w \Phi_\pm,$$

we may write these differentials as

$$\mathrm{Hopf}(\Phi_\pm) = \frac{4 |\partial \Phi_\pm|^2 \bar{\nu}_\pm}{(1 - |w|^2)^2} dw^2,$$

where

$$|\partial \Phi_\pm| = \frac{1 - |w|^2}{1 - |\Phi_\pm|^2} |\partial_w \Phi_\pm|$$

are the corresponding holomorphic energy densities. Since the Hopf differentials are required to add up to zero we have

$$\nu_+ = -\frac{|\partial \Phi_-|^2}{|\partial \Phi_+|^2} \nu_-.$$

Then the area preserving condition for f reduces to

$$\frac{|\partial \Phi_-|^4}{|\partial \Phi_+|^4} |\nu_-|^2 + \frac{|\partial \Phi_-|^2}{|\partial \Phi_+|^2} (1 - |\nu_-|^2) - 1 = 0,$$

which implies

$$\frac{|\partial \Phi_-|^2}{|\partial \Phi_+|^2} = 1,$$

in particular, $\nu_+ = -\nu_-$. The fundamental forms (4.50) of the maximal surface now become

$$I = \frac{4|\partial\Phi_+|^2}{(1-|w|^2)^2}|dw|^2, \qquad I\!I = \frac{\mathrm{i}}{2}\left(\mathrm{Hopf}(\Phi_+) - \overline{\mathrm{Hopf}(\Phi_+)}\right) \tag{4.53}$$

which is clearly just the cotangent bundle description, with conformal factor and quadratic differential given by

$$e^{2\varphi} = \frac{4|\partial\Phi_\pm|^2}{(1-|w|^2)^2}, \qquad q dw^2 = \mathrm{iHopf}(\Phi_+). \tag{4.54}$$

We note that Gauss-Codazzi equations for φ and q now follow directly from the harmonicity (4.41) of $\Phi_\pm$. This gives a nice analytic interpretation of the generalized Mess map $T^*\mathcal{T}(\mathbb{D}) \to \mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ and a proof that is is bijective.

The maps $\Phi_\pm$ are closely related to the pair of diffeomorphisms $\Phi^S_\pm$ constructed in section 2.2 for the description of the Mess map in the compact context, see e.g. [41, 40]. These maps can thus be referred to as the generalized Gauss map, given the resemblance of the construction of the metrics $I_\pm$ with the famous Gauss map between the data on a constant mean curvature surface in $\mathbb{R}^{2,1}$ and hyperbolic metrics.

We note that the realization of $\mathcal{T}(\mathbb{D})$ relevant for all discussions above was the one given by model A. It will be interesting, later on, to understand what type of information the model B realization of $\mathcal{T}(\mathbb{D})$ has to offer in the context of AdS spacetimes. As one might expect from the complex analytic character of model B, this dual realization plays an important role in the holographic description of AdS spacetimes. And, in fact, we shall see in the next chapter that the quadratic differentials coming from the Bers embedding of $\mathcal{T}(\mathbb{D})$ are quite directly related to the quasilocal stress tensor studied in section 3.2.

4.3.4 Chern-Simons connections

To finish the section, we now present a relation between our $\mathcal{T}(\mathbb{D})\times\mathcal{T}(\mathbb{D})$ parametrization of the phase space and Chern-Simons formulation of 2+1 general relativity. We achieve this by a direct computation of the associated pair of flat $PSU(1,1)$ connections to our equidistant AdS metric (4.46) with initial data (4.50). As expected, we shall see that each copy of $\mathcal{T}(\mathbb{D})$ parametrizes one sector of the $PSU(1,1) \times PSU(1,1)$ Chern-Simons theory, thus providing further evidence for the naturality of our construction.

Note that the representation of points in $\mathcal{T}(\mathbb{D})$ by flat $PSU(1,1)$ connections on the unit disc is only meaningful if taken in an appropriate sense. Similarly to the use of hyperbolic metrics as representative of $\mathcal{T}(\mathbb{D})$, we need to differentiate between diffeomorphic connections if they are related by an asymptotically non-trivial diffeomorphism. This is reason for insisting in pulling-back all constructions to the preferred reference disc on $\mathcal{T}(\mathbb{D})$. By doing so, the expressions of the connection components explicitly involve the Beltrami coefficients they represent.

Again, this becomes clearer in the context of nontrivial spatial topology. Let (S,X) and (S,X_μ) denote a reference Riemann surface and its quasiconformal deformation. Then the associated flat $PSU(1,1)$ connections will assign distinct holonomies, respectively A and A_μ, to nontrivial cycles.

It is convenient to start working with isothermal coordinate for the maximal surface

$$ds^2 = -d\tau^2 + \cos^2\tau e^{2\varphi}|dw|^2 + \sin\tau\cos\tau(qdw^2 + \bar{q}d\bar{w}^2) + \sin^2\tau e^{-2\varphi}|q|^2|dw|^2.$$

The frame field and spin connection for the 3-metric above are easily obtained

$$\theta^\tau = d\tau, \qquad \theta^w = e^\varphi \cos\tau dw + e^{-\varphi}\sin\tau\bar{q}d\bar{w}$$

$$\omega^w{}_\tau = -e^\varphi \sin\tau dw + e^{-\varphi}\cos\tau\bar{q}d\bar{w}, \qquad \omega^w{}_w = \partial_w\varphi dw - \partial_{\bar{w}}\varphi d\bar{w},$$

and the corresponding $SL(2,\mathbb{R})$ connections are, in components,

$$A^\pm_w = \frac{1}{2}\begin{bmatrix} \partial_w\varphi & \mp e^\varphi e^{\mp \mathrm{i}\tau} \\ \mathrm{i}e^{-\varphi}e^{\pm\mathrm{i}\tau}q & -\partial_w\varphi \end{bmatrix}, \quad A^\pm_{\bar{w}} = \frac{1}{2}\begin{bmatrix} -\partial_{\bar{w}}\varphi & -\mathrm{i}e^{-\varphi}e^{\mp\mathrm{i}\tau}\bar{q} \\ \mp e^\varphi e^{\pm\mathrm{i}\tau} & \partial_{\bar{w}}\varphi \end{bmatrix},$$
$$A^\pm_\tau = \frac{\mathrm{i}}{2}\begin{bmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{bmatrix}.$$

Here we choose to work with $\mathrm{SU}(1,1)$ generators

$$T_0 = \frac{\mathrm{i}}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_1 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad T_2 = \frac{1}{2}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$$

so that we have

$$\mathrm{tr}\,(T_aT_b) = \frac{1}{2}\eta_{ab}, \qquad [T_a, T_b] = \epsilon_{ab}{}^cT_c.$$

The τ dependence is easily seen to be pure gauge and can be removed by performing a transformation $A_\pm \mapsto g^{-1}\tilde{A}_\pm g + g^{-1}dg$ with

$$g = \begin{bmatrix} e^{\pm\mathrm{i}\tau/2} & 0 \\ 0 & e^{\mp\mathrm{i}\tau/2} \end{bmatrix}.$$

The connections components then become

$$A^\pm_w = \frac{1}{2}\begin{bmatrix} \partial_w\varphi & \mp e^\varphi \\ \mathrm{i}e^{-\varphi}q & -\partial_w\varphi \end{bmatrix}, \quad A^\pm_{\bar{w}} = \frac{1}{2}\begin{bmatrix} -\partial_{\bar{w}}\varphi & -\mathrm{i}e^{-\varphi}\bar{q} \\ \mp e^\varphi & \partial_{\bar{w}}\varphi \end{bmatrix}.$$

Recalling that the Liouville field and the holomorphic quadratic differential can be written, in terms of $\Phi_\pm$, as

$$e^{2\varphi} = \frac{4|\partial\Phi_+|^2}{(1-|w|^2)^2}, \qquad qdw^2 = \mathrm{i}\mathrm{Hopf}(\Phi_+),$$

we need just another gauge transformation $A^{\pm} \mapsto g^{-1}A^{\pm}g + g^{-1}dg$, with

$$g = \begin{bmatrix} (\partial_w \Phi_\pm / |\partial_w \Phi_\pm|)^{-1/2} & 0 \\ 0 & (\partial_w \Phi_\pm / |\partial_w \Phi_\pm|)^{1/2} \end{bmatrix},$$

to see the connections indeed decouple as functions of (μ_+, μ_-). A pull-back to the reference disc then gives

$$\begin{aligned} A_z^{\pm} &= \frac{1}{(1-|f_{\mu_\pm}|^2)} \begin{bmatrix} \frac{1}{2}(\bar{f}_{\mu_\pm}\partial_z f_{\mu_\pm} - f_{\mu_\pm}\partial_z \bar{f}_{\mu_\pm}) & \mp\partial_z f_{\mu_\pm} \\ \mp\partial_z \bar{f}_{\mu_\pm} & -\frac{1}{2}(\bar{f}_{\mu_\pm}\partial_z f_{\mu_\pm} - f_{\mu_\pm}\partial_z \bar{f}_{\mu_\pm}) \end{bmatrix}, \\ A_{\bar{z}}^{\pm} &= \frac{1}{(1-|f_{\mu_\pm}|^2)} \begin{bmatrix} -\frac{1}{2}(\bar{f}_{\mu_\pm}\partial_{\bar{z}} f_{\mu_\pm} - f_{\mu_\pm}\partial_{\bar{z}} \bar{f}_{\mu_\pm}) & \mp\partial_{\bar{z}} \bar{f}_{\mu_\pm} \\ \mp\partial_{\bar{z}} f_{\mu_\pm} & \frac{1}{2}(\bar{f}_{\mu_\pm}\partial_{\bar{z}} f_{\mu_\pm} - f_{\mu_\pm}\partial_{\bar{z}} \bar{f}_{\mu_\pm}) \end{bmatrix}, \end{aligned} \tag{4.55}$$

and we see that each copy of $\mathcal{T}(\mathbb{D})$ parametrizes one of the Chern-Simons sectors, as could have been expected.

Chapter 5

Relations to Holography

We now relate the previous description (2.35) of AdS spacetimes as evolving maximal surface data (4.50) to the Fefferman-Graham description presented in section 3.2. In section 5.1 we start with a description of infinitesimal deformations of the AdS_3 metric generated by both Brown-Henneaux and infinitesimal quasiconformal generators. By interpreting the resulting spacetimes to be nothing but different representations of the same physical state, we are then able to match these generators to leading and subleading order. This shows our quasiconformal deformations are indeed asymptotically non-trivial in the sense of chapter 3 and that, in fact, it includes all Brown-Henneaux excitations on the conformal boundary.

This will then allow us, in section 5.2, to describe the quasilocal stress tensor in terms of the holomorphic quadratic differentials associated to our maximal surface parametrization via the Bers embedding of each copy of $\mathcal{T}(\mathbb{D})$. This can then be used to provide an expression for the spacetime charges in terms of maximal surface data in the $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ parametrization of the phase space. Such description is only valid the infinitesimal level, but we note it admits a natural conjectural generalization for the finite case.

5.1 The infinitesimal case

5.1.1 Infinitesimal Fefferman-Graham metric

In order to perform a comparison between the maximal surface (4.46) and Fefferman-Graham (3.9) descriptions we find convenient to work with coordinates $t = (x^+ + x^-)/2$ and $\theta = (x^+ - x^-)/2$, on the constant radius cylinders, and radial coordinate $\chi = \log(1/\rho)$. The metric (3.14) can then be written as

$$\begin{aligned} ds^2 = \frac{e^{2\chi}}{4}(-dt^2 + d\theta^2) + d\chi^2 + \frac{1}{2}(adt^2 + 2bdtd\theta + ad\theta^2) \\ + \frac{e^{-2\chi}}{4}(a^2 - b^2)(-dt^2 + d\theta^2), \end{aligned} \tag{5.1}$$

where, in comparison with (3.14), $a = a_+ + a_-$ and $b = a_+ - a_-$. In these coordinates, the Brown-Henneaux vector fields (3.23) are given by

$$\begin{aligned} \xi^\chi_{bh} &= -\frac{1}{2}(\partial_+\varepsilon_+ + \partial_-\varepsilon_-) + \cdots \\ \xi^t_{bh} &= \frac{1}{2}(\varepsilon_+ + \varepsilon_-) + e^{-2\chi}(\partial_+^2\varepsilon_+ + \partial_-^2\varepsilon_-) + \cdots \\ \xi^\theta_{bh} &= \frac{1}{2}(\varepsilon_+ - \varepsilon_-) - e^{-2\chi}(\partial_+^2\varepsilon_+ - \partial_-^2\varepsilon_-) + \cdots \end{aligned} \tag{5.2}$$

where $\varepsilon_\pm$ are chiral functions on the conformal boundary and the dots stand for the subleading components.

The infinitesimal version of the metric (5.1) is now given by

$$ds^2 = ds^2_{\mathrm{AdS}_3} + \frac{1}{2}(\delta a dt^2 + 2\delta b dt d\theta + \delta a d\theta^2), \tag{5.3}$$

with the infinitesimal part obtained as the Lie derivative of the AdS_3 metric

$$ds^2_{\mathrm{AdS}_3} = d\chi^2 - \cosh^2\chi dt^2 + \sinh^2\chi d\theta^2,$$

corresponding to $a_+ = a_- = -1/2$ in (5.1), with respect to a Brown-Henneaux vector field (5.2). The arising relation between the perturbations δa and δb and the functions $\varepsilon_\pm$ parametrizing the vector field (5.2) is directly obtained from (3.33) and read

$$\begin{aligned} \delta a &= -(\partial_+\varepsilon_+ + \partial_+^3\varepsilon_+) - (\partial_-\varepsilon_- + \partial_-^3\varepsilon_-), \\ \delta b &= -(\partial_+\varepsilon_+ + \partial_+^3\varepsilon_+) + (\partial_-\varepsilon_- + \partial_-^3\varepsilon_-). \end{aligned} \tag{5.4}$$

5.1.2 Generators of quasiconformal deformations

We now consider a similar infinitesimal description of the spacetime metric (4.46) in terms of initial data (4.50) on a maximal surface. We thus consider a pair $\delta\mu_\pm$ of harmonic Beltrami coefficients and define corresponding infinitesimal quasiconformal deformations of the preferred base disc in $\mathcal{T}(\mathbb{D})$ in the direction of $\delta\mu_\pm$

$$f_{\delta\mu_\pm} = z + \delta z_\pm + \cdots .$$

We remind the reader the first variations $\delta z_\pm$ are solutions of the infinitesimal Beltrami equation

$$\partial_{\bar z}\delta z_\pm = \delta\mu_\pm. \tag{5.5}$$

It then becomes easy to obtain the corresponding infinitesimal versions of the data (4.50)

$$I = \frac{4|dz|^2}{(1-|z|^2)^2} + \frac{2}{(1-|z|^2)^2}\Big[(\delta\bar\mu_+ + \delta\bar\mu_-)dz^2 + (\delta\mu_+ + \delta\mu_-)d\bar z^2\Big]$$

$$I\!I = \frac{\mathrm{i}}{(1-|z|^2)^2}\Big[(\delta\bar\mu_+ - \delta\bar\mu_-)dz^2 - (\delta\mu_+ - \delta\mu_-)d\bar z^2\Big]$$

and, therefore, the infinitesimal AdS metric in equidistant coordinates to a maximal surface

$$ds^2 = ds^2_{\mathrm{AdS}_3} + \frac{2\cos^2\tau}{(1-|z|^2)^2}\Big[(\delta\bar{\mu}_+ + \delta\bar{\mu}_-)dz^2 + (\delta\mu_+ + \delta\mu_-)d\bar{z}^2\Big]$$
$$+\frac{2\mathrm{i}\sin\tau\cos\tau}{(1-|z|^2)^2}\Big[(\delta\bar{\mu}_+ - \delta\bar{\mu}_-)dz^2 - (\delta\mu_+ - \delta\mu_-)d\bar{z}^2\Big]. \tag{5.6}$$

Here,

$$ds^2_{\mathrm{AdS}_3} = -d\tau^2 + \frac{4\cos^2\tau}{(1-|z|^2)^2}|dz|^2 \tag{5.7}$$

is the AdS_3 metric in equidistant coordinates to a geodesic plane.

We would like to compare the infinitesimal metric (5.6), arising in the universal phase space description, with the infinitesimal metric (5.3), obtained from the holographic (Fefferman-Graham) setting. Our interpretation is that (5.6) and (5.3) are simply different coordinate representation of the same physical spacetime. What this means is that both spacetimes are obtained as deformations of AdS_3 by infinitesimal diffeomorphisms whose difference is trivial at the conformal boundary. To see this is indeed the case, the approach we are going to take will be to asymptotically compare the generators of quasiconformal and asymptotic deformations. By performing a change of coordinates, in a neighbourhood of the maximal surface's boundary curve, where both the equidistant and Fefferman-Graham coordinates are valid, we shall see that the generators can be made to agree, to leading and first subleading order, by an identification between the chiral functions $\varepsilon_\pm$ in (5.2) and the pair $u_\pm$ of Zygmund class functions defining (5.6).

We start describing the infinitesimal generator of the metric (5.6). Let's thus consider a general vector field

$$\xi = \xi^\tau\partial_\tau + \xi^z\partial_z + \xi^{\bar{z}}\partial_{\bar{z}}$$

written in the coordinates relevant for the maximal surface description. We would like to fix the components of such vector field so that its action on the AdS_3 metric, via Lie derivative, generates the infinitesimal part of (5.6). We thus take the Lie derivative of $ds^2_{\mathrm{AdS}_3}$ in the direction of ξ

$$\mathcal{L}_\xi ds^2_{\mathrm{AdS}_3} = -2\partial_\tau\xi^\tau d\tau^2 + \left(\frac{4\cos^2\tau\partial_\tau\xi^{\bar{z}}}{(1-|z|^2)^2} - 2\partial_z\xi^\tau\right)d\tau dz$$
$$+\left(\frac{4\cos^2\tau\partial_\tau\xi^z}{(1-|z|^2)^2} - 2\partial_{\bar{z}}\xi^\tau\right)d\tau d\bar{z} + \frac{4\cos^2\tau\partial_z\xi^{\bar{z}}}{(1-|z|^2)^2}dz^2 + \frac{4\cos^2\tau\partial_{\bar{z}}\xi^z}{(1-|z|^2)^2}d\bar{z}^2$$
$$+\frac{4\cos^2\tau}{(1-|z|^2)^2}\left(2\frac{\bar{z}\xi^z + z\xi^{\bar{z}}}{(1-|z|^2)} + \partial_z\xi^z + \partial_{\bar{z}}\xi^{\bar{z}} - 2\tan\tau\xi^\tau\right)|dz|^2$$

and then equate the obtained tensor with the infinitesimal part of the metric (5.6). This now leads to the following set of equations

$$\partial_\tau\xi^\tau = 0, \qquad \frac{4\cos^2\tau\partial_\tau\xi^{\bar{z}}}{(1-|z|^2)^2} - 2\partial_z\xi^\tau = 0,$$
$$2\frac{\bar{z}\xi^z + z\xi^{\bar{z}}}{(1-|z|^2)} + \partial_z\xi^z + \partial_{\bar{z}}\xi^{\bar{z}} - 2\tan\tau\xi^\tau = 0,$$

$$2\partial_{\bar{z}}\xi^z = (1 - \mathrm{i}\tan\tau)\delta\mu_+ + (1 + \mathrm{i}\tan\tau)\delta\mu_-.$$

In view of (5.5), the last equation is clearly satisfied by

$$\xi^z = \frac{1}{2}(1 - \mathrm{i}\tan\tau)\delta z_+ + \frac{1}{2}(1 + \mathrm{i}\tan\tau)\delta z_- = \frac{1}{2}(\delta z_+ + \delta z_-) + \frac{1}{2\mathrm{i}}\tan\tau(\delta z_+ - \delta z_-).$$

The third equation then becomes

$$\begin{aligned} 2\tan\tau\xi^\tau &= \frac{\bar{z}(\delta z_+ + \delta z_-) + z(\delta\bar{z}_+ + \delta\bar{z}_-)}{(1 - |z|^2)} + \frac{1}{2}\partial_z(\delta z_+ + \delta z_-) + \frac{1}{2}\partial_{\bar{z}}(\delta\bar{z}_+ + \delta\bar{z}_-) \\ &+ \tan\tau\left(\frac{1}{\mathrm{i}}\frac{\bar{z}(\delta z_+ - \delta z_-) - z(\delta\bar{z}_+ - \delta\bar{z}_-)}{(1 - |z|^2)} + \frac{1}{2\mathrm{i}}\partial_z(\delta z_+ - \delta z_-) - \frac{1}{2\mathrm{i}}\partial_{\bar{z}}(\delta\bar{z}_+ - \delta\bar{z}_-)\right) \end{aligned}$$

and, using identity (4.36) for each $\delta z_\pm$, we finally have

$$\begin{aligned} \xi^\tau = \frac{1}{2\mathrm{i}}\frac{\bar{z}(\delta z_+ - \delta z_-) - z(\delta\bar{z}_+ - \delta\bar{z}_-)}{(1 - |z|^2)} &+ \frac{1}{4\mathrm{i}}\partial_z(\delta z_+ - \delta z_-) - \frac{1}{4\mathrm{i}}\partial_{\bar{z}}(\delta\bar{z}_+ - \delta\bar{z}_-) \\ &= \frac{1}{\mathrm{i}}\frac{\bar{z}(\delta z_+ - \delta z_-)}{(1 - |z|^2)} + \frac{1}{2\mathrm{i}}\partial_z(\delta z_+ - \delta z_-). \end{aligned}$$

The first and second equations in the set are then directly satisfied.

We have now obtained the infinitesimal quasiconformal generator of the metric (5.6)

$$\begin{aligned} \xi^\tau_{qc} &= \frac{1}{\mathrm{i}}\frac{\bar{z}(\delta z_+ - \delta z_-)}{(1 - |z|^2)} + \frac{1}{2\mathrm{i}}\partial_z(\delta z_+ - \delta z_-), \\ \xi^z_{qc} &= \frac{1}{2}(\delta z_+ + \delta z_-) + \frac{1}{2\mathrm{i}}\tan\tau(\delta z_+ - \delta z_-), \end{aligned} \tag{5.8}$$

which gives us component expressions for what can be interpreted as a Brown-Henneaux vector field in the universal phase space description.

5.1.3 Matching the quasiconformal and asymptotic generators

We now relate the two descriptions using the fact they simply represent different coordinates on the same spacetime. Thus, let us compute the components of the Brown-Henneaux vector fields (5.2) in the coordinates used in (5.7). The coordinate transformation relating the AdS_3 metric in the form (5.1) and its equidistant coordinates description is given by

$$\tan t = \frac{1 - |z|^2}{1 + |z|^2}\tan\tau, \quad \sinh\chi = \frac{2|z|}{1 - |z|^2}\cos\tau, \quad \theta = \arg z.$$

These can be directly applied to the Brown-Henneaux vector field (5.2), which leads to the following relation between their coordinate components

$$\begin{aligned} \xi^\tau_{bh} &= \frac{1 + |z|^2}{1 - |z|^2}\xi^t_{bh} + \frac{2|z|\sin\tau}{[(1 - |z|^2)^2 + 4|z|^2\cos^2\tau]^{1/2}}\xi^\chi_{bh} \\ \xi^w_{bh} &= z\tan\tau\xi^t_{bh} + \mathrm{i}z\xi^\theta_{bh} - \frac{1}{2}\frac{z}{|z|}\frac{(1 - |z|^4)\sec\tau}{[(1 - |z|^2)^2 + 4|z|^2\cos^2\tau]^{1/2}}\xi^\chi_{bh}. \end{aligned} \tag{5.9}$$

We need only consider the asymptotic behaviour of these components around the $\tau = 0$ maximal surface. Thus, expanding to first order in τ, we get the following leading and first subleading terms

$$\xi^\tau_{bh} = \Big(\frac{1}{1 - |z|^2} - \frac{1}{2}\Big)(\varepsilon_+ + \varepsilon_-)\big|_{\tau=0} + \cdots$$

$$\xi^z_{bh} = \frac{1}{2}\mathrm{i}z\Big((\varepsilon_+ - \varepsilon_-)\big|_{\tau=0} + \frac{(1-|z|^2)}{2\mathrm{i}}(\partial_+\varepsilon_+ + \partial_-\varepsilon_-)\big|_{\tau=0}\Big)$$
$$+\frac{\tau}{2\mathrm{i}}\mathrm{i}z\Big((\varepsilon_+ + \varepsilon_-)\big|_{\tau=0} - \frac{1-|z|^2}{2\mathrm{i}}(\partial_+\varepsilon_+ - \partial_-\varepsilon_-)\big|_{\tau=0}\Big) + \cdots . \tag{5.10}$$

It now becomes clear that these are nothing but the leading components of the quasiconformal generators (5.8) with

$$\delta z_\pm = \pm\mathrm{i}z\Big(\varepsilon_\pm\big|_{\tau=0} \pm \frac{(1-|z|^2)}{2\mathrm{i}}\partial_\mp\varepsilon_\mp\big|_{\tau=0}\Big) + \cdots . \tag{5.11}$$

The restriction to the boundary of the above relation can be readily recognized as the relation (4.18) between infinitesimal quasiconformal maps and the corresponding Zygmund class functions on the circle

$$\frac{\delta z_\pm(e^{\mathrm{i}\theta})}{\mathrm{i}e^{\mathrm{i}\theta}} = u_\pm(e^{\mathrm{i}\theta}) = \pm\varepsilon_\pm(\pm\theta). \tag{5.12}$$

We have therefore identified the space of chiral Brown-Henneaux generators with the tangent space to each $\mathcal{T}(\mathbb{D})$ sector in the universal phase space. At least for infinitesimal metrics, this shows how one may continue the geometry of the maximal surface's domain of dependence beyond the Cauchy horizon. Indeed, by asymptotically matching the quasiconformal and asymptotic generators, we have shown the infinitesimal metrics (5.6) and (5.3) are related by an asymptotically trivial diffeomorphism and, therefore, they represent the same physical state.

5.2 Bers embedding and the stress tensor

5.2.1 Analytic continuation

We now develop an interpretation of the obtained relation (5.12) between the universal phase space and the holographic descriptions in terms of the B model realization of universal Teichmüller space. In the previous section we have seen how the two chiral functions $\varepsilon_\pm$ parametrizing the Brown-Henneaux vector fields are given in terms of Zygmund functions $u_\pm$ parametrizing the boundary values of the (infinitesimal) quasiconformal maps. An equally interesting question we would now like to address is that of a relation between the holographic stress-energy tensor components — functions a, b in (5.1) — and these quasiconformal maps.

In this section we shall see that this relation is that between the holomorphic quadratic differentials arising via the Bers embedding of $\mathcal{T}(\mathbb{D})$. In other words, we shall see that the stress-energy tensor components of the holographic description are nothing but the components of the quadratic differentials arising from the B model realization of $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$. Here, our convention for the dual realization of $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ will be to use the B-model for the first copy of $\mathcal{T}(\mathbb{D})$ and the $\hat{\mathrm{B}}$-model for the second. Thus a point $([\mu_+], [\mu_-]) \in \mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ gives rise to a pair $(h_+, h_-) \in \mathrm{HQD}(\mathbb{D}) \times \overline{\mathrm{HQD}(\mathbb{D})}$ formed by holomorphic and anti-holomorphic quadratic differentials on $\mathbb{D}^*$.

We shall continue to work on the infinitesimal case, but note that the results we shall obtain in this section presents a natural conjectural extension to the general case. The clear advantage of working with the infinitesimal description are the explicit expressions for the Laurent coefficients of $h_\pm$ in terms of the Zygmund functions $u_\pm$ and, therefore, in terms of the Brown-Henneaux generators. We remind the reader that, by expanding the quadratic differentials in Laurent series

$$h_+ = \sum_{k\geq 2} \frac{h^+_{k-2}}{z^{k+2}}, \qquad h_- = \sum_{k\geq 2} \frac{h^-_{k-2}}{\bar{z}^{k+2}}$$

and the corresponding Zygmund functions in Fourier series

$$u_+ = \sum_{k\geq 2} u^+_k e^{\mathrm{i}k\theta}, \qquad u_- = \sum_{k\geq 2} u^-_k e^{\mathrm{i}k\theta},$$

we have obtained in 4.1 the following relations

$$h^\pm_{k-2} = \pm\mathrm{i}\bar{u}^\pm_k (k - k^3). \tag{5.13}$$

Turning to the holographic description, we now rewrite the quasilocal stress tensor

$$T = adt^2 + 2bdtd\theta + ad\theta^2 \tag{5.14}$$

of section 3.2 in a more suggestive way. To this end, we shall analytically continue the t coordinate to take imaginary values. Thus, let us continue all the functions appearing in T via

$$t = \frac{1}{2\mathrm{i}} \log |z|^2, \qquad \theta = \frac{1}{2\mathrm{i}} \log \frac{z}{\bar{z}}, \tag{5.15}$$

so that the new (imaginary) time coordinate runs between $-\mathrm{i}\infty$ and $\mathrm{i}\infty$ while z runs over the complex plane, with the unit circle $|z| = 1$ corresponding to $t = 0$. With this choice, we have

$$e^{\mathrm{i}(t+\theta)} = z, \qquad e^{\mathrm{i}(t-\theta)} = \bar{z},$$

so that functions of $e^{\mathrm{i}(t\pm\theta)}$ become holomorphic (anti-holomorphic) functions on the complex plane. In particular, the functions $a_\pm$, whose sum and difference give a, b, would now seem to become a holomorphic and anti-holomorphic function on the complex plane.

Let us expand $a_\pm$ into Fourier modes. When restricted to $t = 0$ these are periodic functions of θ, and so the Fourier expansion is indeed possible. We have

$$a_\pm(t \pm \theta) = \sum_{k=-\infty}^{\infty} a^\pm_k e^{\mathrm{i}k(t\pm\theta)},$$

with $\bar{a}^\pm_k = a^\pm_{-k}$, imposed by the fact these are real functions. Now, with hindsight, we shall not continue a_+ and a_- as holomorphic or anti-holomorphic functions on the whole complex plane. The continuation we shall perform will be to take, say, the negative frequency part of $a_\pm$ and continue this part only as holomorphic/anti-holomorphic functions on the complement of the disc. The reality condition $\bar{a}^\pm_k = a^\pm_{-k}$ then ensures there is no loss of information in this procedure, since we may obtain the positive frequency part by complex conjugation on the unit circle.

Thus, let us introduce

$$\tilde{a}_\pm(t \pm \theta) = -\frac{1}{2} + \sum_{k=-\infty}^{-2} a_k^\pm e^{\mathrm{i}k(t\pm\theta)}$$

for the negative frequency parts. Here we have used the fact that $|k| \geq 2$ for the expansion of the variation of the $a_\pm$ components around AdS_3, which will become manifest below. We now continue the negative frequency parts, via (5.15), to the complement of the unit disc to get

$$\tilde{a}_+(z) = -\frac{1}{2} + \frac{a_{-2}^+}{z^2} + \frac{a_{-3}^+}{z^3} + \cdots, \qquad \tilde{a}_-(\bar{z}) = -\frac{1}{2} + \frac{a_{-2}^-}{\bar{z}^2} + \frac{a_{-3}^-}{\bar{z}^3} + \cdots, \tag{5.16}$$

which are, respectively, holomorphic and anti-holomorphic functions on $\mathbb{D}^*$. We can now analytically continue the negative frequency part $\tilde{T}$ of the stress tensor (5.14), which is the tensor T with functions $a_\pm$ replaced by their negative frequency parts. A simple computation gives

$$\tilde{T} = -\frac{\tilde{a}_+(z)}{z^2}dz^2 - \frac{\tilde{a}_-(\bar{z})}{\bar{z}^2}d\bar{z}^2. \tag{5.17}$$

The last step we shall need, before relating the negative frequency parts $\tilde{a}_\pm$ of the stress-energy tensor to the quadratic differentials arising via the Bers embedding, is to obtain the relation between $a_k^\pm$ and the Fourier coefficients of the parametrizing functions $\varepsilon_\pm$ of the Brown-Henneaux generators. From (5.4) we know that

$$a_\pm = -\frac{1}{2} - \partial_\pm \varepsilon_\pm - \partial_\pm^3 \varepsilon_\pm.$$

Thus, if we expand

$$\varepsilon_\pm(t \pm \theta) = \sum_{k=-\infty}^{\infty} \varepsilon_k^\pm e^{\mathrm{i}k(t\pm\theta)},$$

with $\varepsilon_{-k}^\pm = \bar{\varepsilon}_k^\pm$, we get the following relation between the Fourier coefficients

$$a_0^\pm = -\frac{1}{2}, \qquad a_k^\pm = -\mathrm{i}\varepsilon_k^\pm(k - k^3), \quad k \geq 2.$$

The relation to the quadratic differentials $h_\pm$ are now easily obtained. Using (5.12), we may write

$$a_{-k}^\pm = \pm\mathrm{i}\bar{u}_k^\pm(k - k^3), \quad k \geq 2$$

and, comparing this with (5.13), this gives a very simple relation,

$$h_{k-2}^\pm = a_{-k}^\pm, \qquad k \geq 2,$$

between the Laurent coefficients in the expansions (5.16) of the negative frequency parts of the stress-energy tensor and those of the Bers quadratic differentials (4.20), (4.32).

The relation above can also be written as a direct relation

$$\tilde{a}_+(z) = -\frac{1}{2} + z^2 h^+(z), \qquad \tilde{a}_-(\bar{z}) = -\frac{1}{2} + \bar{z}^2 h^-(\bar{z}). \tag{5.18}$$

And, finally, the analytic continuation of the negative frequency part (5.17) of the stress-energy tensor is equal to (minus) the sum of two quadratic differentials arising via the Bers embedding

$$\tilde{T} = \Big(\frac{1}{2z^2} - h^+(z)\Big)dz^2 + \Big(\frac{1}{2\bar{z}^2} - h^-(\bar{z})\Big)d\bar{z}^2 \tag{5.19}$$

which is our final result for the (infinitesimal) relation between the maximal surface and the holographic descriptions. Note that the constant term

$$\tilde{T}_{\mathrm{AdS}_3} = \frac{1}{2z^2}dz^2 + \frac{1}{2\bar{z}^2}d\bar{z}^2 \tag{5.20}$$

is simply the negative frequency part of the reference spacetime, here taken to be AdS_3.

5.2.2 Spacetime charges

We finish this section with an expression for the conserved charges of an AdS spacetime in terms of the maximal surface parametrization. In section 3.2, we have obtained these charges as spatial boundary integrals of the components of the quasilocal stress tensor

$$Q[\xi] = \frac{1}{2\pi}\int_{\partial S} d\theta \Big[a_+\xi^+ + a_-\xi^-\Big].$$

Now, the relations above allows for a direct translation to the maximal surface description. The full components $a_\pm$ on the circle can be obtained from their negative frequency parts by adding their complex conjugate. Thus, we have $2\mathrm{Re}(\tilde{a}_\pm)\Big|_{|z|=1} = a_\pm(\pm\theta)$ and, therefore,

$$Q[\xi] = \frac{1}{2\pi}\int_{\partial S} d\theta \Big[2\mathrm{Re}(-\frac{1}{2} + e^{2\mathrm{i}\theta}h_+)\xi^+ + 2\mathrm{Re}(-\frac{1}{2} + e^{-2\mathrm{i}\theta}h_-)\xi^-\Big]. \tag{5.21}$$

Decomposing the Brown-Henneaux generators into modes, see section 3.2, we obtain

$$Q[\xi^\pm_n] = \frac{1}{2\pi}\int_{\partial S} d\theta \Big(-\frac{1}{2} + \sum_{k\geq 2} h^\pm_{k-2}e^{\mp \mathrm{i}k\theta} + \bar{h}^\pm_{k-2}e^{\pm \mathrm{i}k\theta}\Big)e^{\mp \mathrm{i}n\theta} = \begin{cases} -\frac{1}{2}, & n=0 \\ \bar{h}^\pm_{n-2}, & n\geq 2 \\ h^\pm_{-n-2}, & n\leq -2. \end{cases}$$

In particular

$$M = \frac{1}{2}(Q[\xi^+_0] + Q[\xi^-_0]) = -1, \qquad J = \frac{1}{2}(Q[\xi^+_0] - Q[\xi^-_0]) = 0$$

as could have been expected since there are no $1/z^2$ and $1/\bar{z}^2$ terms in the expansion of the infinitesimal quadratic differentials $h^\pm$, see (4.20), (4.32). This only says, the first order variations of mass and angular momentum, computed at the reference spacetime AdS_3, are zero for any infinitesimal deformation. It is however clear that by considering non-trivial spatial topologies the expressions above will render non-trivial charges in each asymptotic region of more general spacetimes.

As a final remark, we note that, although the formulas above are, of course, only valid at the infinitesimal level, they admit a natural conjectural extension to the finite case, by the understanding the Bers embedding quadratic differentials $h_{\pm}$ are the analytic continuations of the negative frequency parts of $a_{\pm}$ composing the stress-energy tensor, see (5.18). It is then natural to conjecture that such negative frequency components continue to be related to the Bers quadratic differentials in the same way as they do in the infinitesimal case. This would leads to a more explicit relation between the universal phase space and the holographic (Fefferman-Graham) descriptions, where one could directly construct from holographic data the whole of spacetime's bulk geometry. We leave an attempt to demonstrate such conjectural extension to the finite case to future work.

Chapter 6

Symplectic Structure

In this chapter, we would like to address two points. First we would like to obtain the relation between the natural symplectic structures on $T^*\mathcal{T}(\mathbb{D})$ and $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ parametrizing the universal phase space and the symplectic structure induced by the gravitational action. We shall see, up to boundary terms, the induced gravitational symplectic form obtained from the (ADM) formulation of gravity will provide the canonical cotangent bundle symplectic form on the $T^*\mathcal{T}(\mathbb{D})$ parametrization. On the other hand, the symplectic form coming from the Chern-Simons formulation will describe the difference of Weil-Petersson symplectic forms on $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$.

This lead us to the second point we want to address, namely the symplectic properties of the map Mess : $T^*\mathcal{T}(\mathbb{D}) \to \mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$. It is clear that the possibility of rewriting the Einstein-Hilbert action in both ADM and CS forms suggests this map is a symplectomorphism. We shall verify this explicitly by considering the pull-back of the $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ symplectic forms by the Mess map and showing it agrees with the cotangent bundle symplectic form.

We warn the reader the results of this chapter are not yet completely developed and are still under research, in collaboration with Jean-Marc Schlenker. In particular, we shall not consider the delicate issues associated with boundary terms. The results presented here can then be seen as the first necessary steps for a proof that Mess : $T^*\mathcal{T}(\mathbb{D}) \to \mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ is symplectic. We describe these partial results for we feel most the ingredients for a complete proof are already present, only some technical difficulties remain to be addressed.

6.1 Symplectic structures on $T^*\mathcal{T}(\mathbb{D})$ and $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$

6.1.1 The horizontal directions on $T_qT^*\mathcal{T}(\mathbb{D})$

In section 4.1 we have given a description of the tangent space to (universal) Teichmüller space $T_{[0]}\mathcal{T}(\mathbb{D})$ as the space of harmonic Beltrami coefficients $\delta\mu$ on $\mathbb{D}$. At an arbitrary point $[\mu] \in \mathcal{T}(\mathbb{D})$, we have seen that $T_{[\mu]}\mathcal{T}(\mathbb{D}) = R_{[\mu]}T_{[0]}\mathcal{T}(\mathbb{D})$ is simply obtained by right translating the tangent space at the origin by $[\mu]$. We remind the reader that a tangent vector $\delta\mu$ at point $[\mu] \in \mathcal{T}(\mathbb{D})$ then defines a one-parameter family of quasiconformal maps

$$f_{\mu+t\delta\mu}(z) = f_{t\delta\tilde{\mu}} \circ f_\mu(z) = f_\mu(z) + t\delta w \circ f_\mu(z) + O(t^2) \tag{6.1}$$

with

$$\partial_{\bar{z}}(\delta w \circ f_\mu) - \mu\partial_z(\delta w \circ f_\mu) = \delta\mu\partial_z f_\mu \tag{6.2}$$

or, equivalently,

$$\partial_{\bar{w}}\delta w = \delta\tilde{\mu}, \qquad \delta\tilde{\mu} \circ f_\mu = \frac{\delta\mu}{(1-|\mu|^2)}\frac{\partial_z f_\mu}{\partial_{\bar{z}}\bar{f}_\mu}.$$

We therefore already have an understanding about the tangent bundle over our phase space in its parametrization by $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$. For the discussion that follows, we shall also need a similar understanding of this tangent bundle in the $T^*\mathcal{T}(\mathbb{D})$ parametrization.

As for the tangent bundle over any vector bundle, $TT^*\mathcal{T}(\mathbb{D})$ admits a decomposition into vertical and horizontal sub-bundles. The vertical sub-bundle is, as usual, canonically defined. It is given, at each point $([\mu], q) \in T^*\mathcal{T}(\mathbb{D})$, as the kernel subspace of the derivative of the projection map $\pi : T^*\mathcal{T}(\mathbb{D}) \to \mathcal{T}(\mathbb{D})$, associating to $([\mu], q)$ the conformal structure $[\mu] = \pi([\mu], q)$ with respect to which q is holomorphic. Thus, at each point $([\mu], q)$ the vertical subspace consists of infinitesimal quadratic differentials δq holomorphic with respect to $[\mu]$.

The horizontal sub-bundle, on the other hand, is not canonical. Its definition amounts to the choice of a connection on $T^*\mathcal{T}(\mathbb{D})$ identifying fibres $T^*_{[\mu]}\mathcal{T}(\mathbb{D})$ at distinct base points via parallel transport. Here we shall not delve into the theory of connections over (infinite dimensional) vector bundles and will be satisfied with a simple, rather natural, choice for parallel transporting holomorphic quadratic differentials on $\mathbb{D}$ along quasiconformal deformations.

Given a quadratic differential q holomorphic with respect to $[\mu] \in \mathcal{T}(\mathbb{D})$ and a curve $[\mu_t]$ in $\mathcal{T}(\mathbb{D})$ with $\mu_0 = \mu$, we define the parallel transport of q along $[\mu_t]$ by applying the the quasiconformal map $f_{\mu_t} \circ f_\mu^{-1}$ to its argument

$$q(w) \longmapsto q(w_t) = q(f_{\mu_t} \circ f_\mu^{-1}(w)).$$

We are thus simply substituting in q the holomorphic coordinate w, associated with $[\mu] \in \mathcal{T}(\mathbb{D})$, with the holomorphic coordinate w_t, associated with $[\mu_t]$. Note this is not the same as taking the pull-back of the quadratic differential which would clearly not render a holomorphic differential with respect to $[\mu_t]$.

To define the horizontal sub-bundle let's now take a tangent vector $\delta\mu \in T_{[\mu]}\mathcal{T}(\mathbb{D})$ and consider the one-parameter family (6.1). Our definition for the parallel transport of the holomorphic quadratic differential q then reads

$$q(w) \longmapsto q(w) + t\partial_w q\delta w + O(t^2),$$

which describes a horizontal vector in $T_{([\mu],q)}T^*\mathcal{T}(\mathbb{D})$. We may thus write

$$H_{([\mu],q)}T^*\mathcal{T}(\mathbb{D}) = \Big\{(\delta\mu, \partial_w q\delta w); \delta\mu \in T_{[\mu]}\mathcal{T}(\mathbb{D})\Big\}$$

for the horizontal subspace and, similarly,

$$V_{([\mu],q)}T^*\mathcal{T}(\mathbb{D}) = \Big\{(0, \delta q); \delta q \in T^*_{[\mu]}\mathcal{T}(\mathbb{D})\Big\}$$

for the vertical subspace.

It should be clear that the quadratic differential $q + t\partial_w q\delta w$ is now holomorphic with respect to the conformal structure $[\mu + t\delta\mu]$ so our definition does makes sense, at least for the case of trivial topology. To see this we only need the variations, in the $\delta\mu$ direction, of the coordinate vector fields ∂_w and $\partial_{\bar{w}}$. Let's start with the variations of the coordinate 1-forms dw and $d\bar{w}$

$$\delta_{\delta\mu} dw = d\delta w = \partial_w \delta w dw + \delta\tilde{\mu} d\bar{w}, \qquad \delta_{\delta\mu} d\bar{w} = d\delta\bar{w} = \partial_{\bar{w}}\delta\bar{w} d\bar{w} + \delta\bar{\tilde{\mu}} dw. \tag{6.3}$$

The variations $\delta\partial_w$ and $\delta\partial_{\bar{w}}$ are the obtained by imposing

$$(dw + \delta_{\delta\mu} dw)(\partial_w + \delta_{\delta\mu}\partial_w) = 1, \qquad (dw + \delta_{\delta\mu} dw)(\partial_{\bar{w}} + \delta_{\delta\mu}\partial_{\bar{w}}) = 0,$$

which is easily seen to provide

$$\delta_{\delta\mu}\partial_w = -\partial_w \delta w \partial_w - \delta\bar{\tilde{\mu}}\partial_{\bar{w}}, \qquad \delta_{\delta\mu}\partial_{\bar{w}} = -\partial_{\bar{w}}\delta\bar{w}\partial_{\bar{w}} - \delta\mu\partial_w.$$

It is now a simple computation that $q + t\partial_w q\delta z$ satisfies

$$(\partial_{\bar{w}} + \delta\partial_{\bar{w}})(q + t\partial_w q\delta w) = \partial_{\bar{w}} q + t\Big(\partial_{\bar{w}}(\partial_w q\delta w) - \partial_{\bar{w}}\delta\bar{w}\partial_{\bar{w}} q - \delta\tilde{\mu}\partial_w q\Big) = 0,$$

where we have made use of the holomorphicity of q.

With the above definitions for the vertical and horizontal subspaces we may now write down the action of a tangent vector $(\delta\mu, \delta q) \in T_{([\mu],q)}T^*\mathcal{T}(\mathbb{D})$ on an arbitrary function $\mathcal{Q}$ on $T^*\mathcal{T}(\mathbb{D})$

$$\delta_{(\delta\mu,\delta q)}\mathcal{Q} = \partial_w \mathcal{Q}\delta w + \partial_{\bar{w}}\mathcal{Q}\delta\bar{w} + \frac{\delta\mathcal{Q}}{\delta q}\delta q + \frac{\delta\mathcal{Q}}{\delta\bar{q}}\delta\bar{q}.$$

In particular, the action of such tangent vector on the quadratic differential q it self becomes

$$\delta_{(\delta\mu,\delta q)} q = \partial_w q\delta w + \delta q.$$

6.1.2 The cotangent bundle symplectic form

In sections 2.2 and 4.3 we gave distinct parametrizations of the space of 2+1 dimensional AdS spacetimes, with both compact and noncompact spatial topology, in terms of Teichmüller space of an embedded maximal surface. We note that both parametrizations carry natural symplectic structures. The first, being a cotangent bundle $T^*\mathcal{T}(\mathbb{D})$, carries the canonical cotangent bundle symplectic form. The second, given by two copies of $\mathcal{T}(\mathbb{D})$, carries the structure induced from the

Weil-Petersson symplectic form in each copy of $\mathcal{T}(\mathbb{D})$. The constructions given in the previous chapters also provided a bijective map between $T^*\mathcal{T}(\mathbb{D})$ and $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$, referred to here as the (generalized) Mess map. The existence of natural symplectic structures in both spaces, and a bijection between them then leads to the question of whether the Mess map is natural from the symplectic point of view, that is, if its action maps one symplectic structure to the other. We shall now describe some evidence suggesting this is indeed the case. A proof will be given at the end of the section.

Thus, let us now consider the induced symplectic structures coming from the gravitational Einstein-Hilbert action (2.12). In the ADM formulation of 2+1 gravity, see section 2.1, the gravitational pre-symplectic form is obtained directly from this action in its Hamiltonian form

$$\Omega^{GR} = \int_S d^2x \, \mathrm{tr}\,(\delta\Pi \wedge \delta I). \tag{6.4}$$

Note that, since our considerations allows for non spatially compact spacetimes, the Einstein-Hilbert action is complemented by the York-Gibbons-Hawking boundary term and renormalization counter term (3.15). These terms do contribute the symplectic potential, which is obtained from the first variation of the Lagrangian of the theory, but they do not alter the symplectic structure since they only add to the symplectic potential a variational exact form. Another source of ambiguity is the possibility of adding (spacetime) exact terms to the symplectic potential itself. These terms would then contribute to the symplectic structure, since they need not be exact in the variational sense. It may be that the asymptotic fall-off conditions are enough to ensure these contributions are zero or, else, to fix them to particular values, see section 3 of [69] for similar discussion in the case of asymptotically flat spacetimes. We shall not address such issues leaving them to future studies. In fact, we shall discard all boundary terms in the considerations that follow. Some comments on such terms will be given in the end of the section.

We remind the reader, Ω^{GR} is the canonical symplectic structure on the cotangent bundle of the space of two dimensional Riemann metrics on S. This is not yet the reduced (physical) phase space of 2+1 gravity and we still need to impose the Gauss-Codazzi constraints and eliminate the remaining gauge freedom. To obtain the physical gravitational symplectic structure on the reduced phase space parametrized by $T^*\mathcal{T}(\mathbb{D})$, we shall thus impose the constraints and gauge fixing conditions directly by writing the canonical fields explicitly in terms of the variable $([\mu], q) \in T^*\mathcal{T}(\mathbb{D})$.

Since we may use any Cauchy surface S in the computation of the symplectic form, we make the simplest choice of working with the maximal surface data

$$I = e^{2\varphi}|dw|^2, \qquad I\!I = \frac{1}{2}(q dw^2 + \bar{q} d\bar{w}^2).$$

The canonically conjugated momentum can be easily computed as in section 2.2

$$\Pi = \frac{\sqrt{I}}{2\pi}(I^{-1} I\!I I^{-1} - \mathrm{tr}\,(I^{-1} I\!I) I^{-1}) = \frac{e^{-2\varphi}}{\pi}(\bar{q}\partial_w^2 + q\partial_{\bar{w}}^2),$$

and we may compute the variations of (I, Π) in the direction of $(\delta\mu, \delta q)$

$$\delta_{(\delta\mu,\delta q)} I = e^{2\varphi}\Big(\delta\bar{\tilde{\mu}} dw^2 + \delta\tilde{\mu} d\bar{w}^2 + (2\delta_{(\delta\mu,\delta q)}\varphi + \partial_w \delta w + \partial_{\bar{w}}\delta w)|dz|^2\Big),$$

$$\begin{aligned}\delta_{(\delta\mu,\delta q)}\Pi &= \sqrt{I}\delta_{(\delta\mu,\delta q)}\Big(\frac{\Pi}{\sqrt{I}}\Big) + \frac{1}{2}\mathrm{tr}\,(I^{-1}\delta_{(\delta\mu,\delta q)} I)\Pi \\ = \frac{e^{-2\varphi}}{\pi}\Big(&- 2(q\delta\tilde{\mu} + \bar{q}\delta\bar{\tilde{\mu}})\partial_w\partial_{\bar{w}} + (\delta_{(\delta\mu,\delta q)}\bar{q} - \bar{q}(2\delta_{(\delta\mu,\delta q)}\varphi + \partial_w\delta w - \partial_{\bar{w}}\delta\bar{w}))\partial_w^2 \\ &+ (\delta_{(\delta\mu,\delta q)} q - q(2\delta_{(\delta\mu,\delta q)}\varphi - \partial_w\delta w + \partial_{\bar{w}}\delta\bar{w}))\partial_{\bar{w}}^2\Big).\end{aligned}$$

Note that, when taking these variations, one must take into account that the canonical momentum Π is not a tensor but a tensor density weight 1.

Now, substituting the above expressions in (6.4), and for now discarding boundary terms, the gravitational pre-symplectic structure becomes simply

$$\begin{aligned}\Omega^{GR} &= \frac{1}{2\pi\mathrm{i}}\int_{f_\mu(\mathbb{D})} dw \wedge d\bar{w}\Big(\delta_{(\delta\mu,\delta q)} q \wedge \delta\tilde{\mu} + \delta_{(\delta\mu,\delta q)}\bar{q} \wedge \delta\bar{\tilde{\mu}} + 2\bar{q}\partial_{\bar{w}}\delta\bar{w} \wedge \delta\bar{\tilde{\mu}} + 2q\partial_w\delta w \wedge \delta\tilde{\mu}\Big) \\ &= \frac{1}{2\pi\mathrm{i}}\int_{f_\mu(\mathbb{D})} dw \wedge d\bar{w}\Big(\delta q \wedge \delta\tilde{\mu} + \delta\bar{q} \wedge \delta\bar{\tilde{\mu}}\Big) + \frac{1}{2\pi\mathrm{i}}\int_{\partial f_\mu(\mathbb{D})}\Big(\bar{q}\delta\bar{w} \wedge d\delta\bar{w} - q\delta w \wedge d\delta w\Big) \\ &= \frac{1}{2\pi\mathrm{i}}\int_{f_\mu(\mathbb{D})} dw \wedge d\bar{w}\Big(\delta q \wedge \delta\tilde{\mu} + \delta\bar{q} \wedge \delta\bar{\tilde{\mu}}\Big) = \frac{1}{\pi}\Omega^{T^*\mathcal{T}},\end{aligned} \quad (6.5)$$

which gives, up to a multiplicative factor, the cotangent bundle symplectic structure on $T^*\mathcal{T}(\mathbb{D})$.

6.1.3 The Chern-Simons symplectic form

From the Chern-Simons theory point of view the symplectic structure is given by a different expression. For a single $PSU(1,1)$ Chern-Simons theory, the Hamiltonian formalism gives the following symplectic form on the space of all connections over the spatial slice S

$$\Omega^{CS} = \frac{k}{4\pi}\int_S \mathrm{tr}\,(\delta A \wedge \delta A). \quad (6.6)$$

This is to be restricted to connections satisfying the flatness condition

$$F[A] = dA + A \wedge A = 0$$

and one must further mod out the remaining gauge degrees of freedom. Once again, we shall impose the flatness constraint by working directly with the reduced phase space parametrization by $\mathcal{T}(\mathbb{D})$. Note that we are only interested in the geometric sector of the theory. Therefore, we do not consider connections with no metric interpretation.

We shall now start working in the reference point $[0] \in T(\mathbb{D})$. The connection can then be written

$$A = \frac{1}{(1-|z|^2)}\begin{bmatrix}\frac{1}{2}(\bar{z}dz - zd\bar{z}) & -dz \\ -d\bar{z} & -\frac{1}{2}(\bar{z}dz - zd\bar{z})\end{bmatrix},$$

and we now compute its variation in the direction of a tangent vector $\delta\mu$

$$\delta_{\delta\mu} A_z = \frac{1}{2}\begin{bmatrix} -\frac{1}{2}\partial_z\Big(\partial_z\delta z - \partial_{\bar z}\delta\bar z\Big) & -\frac{\partial_z\delta z - \partial_{\bar z}\delta\bar z}{(1-|z|^2)} \\ -\frac{2\delta\bar\mu}{(1-|z|^2)} & \frac{1}{2}\partial_z\Big(\partial_z\delta z - \partial_{\bar z}\delta\bar z\Big)\end{bmatrix}$$

$$\delta_{\delta\mu} A_{\bar z} = \frac{1}{2}\begin{bmatrix} -\frac{1}{2}\partial_{\bar z}\Big(\partial_z\delta z - \partial_{\bar z}\delta\bar z\Big) & -\frac{2\delta\mu}{(1-|z|^2)} \\ \frac{\partial_z\delta z - \partial_{\bar z}\delta\bar z}{(1-|z|^2)} & \frac{1}{2}\partial_{\bar z}\Big(\partial_z\delta z - \partial_{\bar z}\delta\bar z\Big)\end{bmatrix}.$$

Here, we have made use of the identities (4.35) and (4.36) of section 4.1

$$\partial_z\delta\mu + 2\frac{\bar z\delta\mu}{(1-|z|^2)} = 0, \qquad 2\frac{\bar z\delta z + z\delta\bar z}{(1-|z|^2)} + \partial_z\delta z + \partial_{\bar z}\delta\bar z = 0.$$

We may now introduce these expressions in (6.6) to obtain, again discarding boundary terms, the following expression

$$\begin{aligned}\Omega^{CS} &= \frac{k}{8\pi}\int_{\mathbb{D}} dz\wedge d\bar z\frac{\delta\bar\mu\wedge\delta\mu}{(1-|z|^2)^2} + \frac{k}{4\pi}\int_{\partial\mathbb{D}}(\partial_z\delta z - \partial_{\bar z}\delta\bar z)\wedge d(\partial_z\delta z - \partial_{\bar z}\delta\bar z) \\ &= \frac{k}{8\pi}\int_{\mathbb{D}} dz\wedge d\bar z\frac{\delta\bar\mu\wedge\delta\mu}{(1-|z|^2)^2} = \frac{k}{8\pi}\Omega^{WP},\end{aligned} \tag{6.7}$$

which is proportional to the Weil-Petersson symplectic structure on $\mathcal{T}(\mathbb{D})$.

Thus, the gravitational pre-symplectic structure (at the reference point) in the Chern-Simons formulation is now simply obtained as the difference of two Weil-Petersson symplectic forms

$$\Omega^{CS}_+ - \Omega^{CS}_- = \frac{k}{8\pi}\int_{\mathbb{D}}\frac{dz\wedge d\bar z}{(1-|z|^2)^2}\Big(\delta\bar\mu_+\wedge\delta\mu_+ - \delta\bar\mu_-\wedge\delta\mu_-\Big),$$

thus suggesting the generalized Mess map $T^*\mathcal{T}(\mathbb{D}) \to \mathcal{T}(\mathbb{D})\times\mathcal{T}(\mathbb{D})$ is in fact symplectic.

6.2 Symplectic properties of the Mess map

6.2.1 The derivative of the map Mess

We now turn to a more explicit argument showing Mess : $T^*\mathcal{T}(\mathbb{D}) \to \mathcal{T}(\mathbb{D})\times\mathcal{T}(\mathbb{D})$ is a symplectomorphism. Our approach will be to consider, at an arbitrary point $([\mu_+],[\mu_-]) \in \mathcal{T}(\mathbb{D})\times\mathcal{T}(\mathbb{D})$, the pull-back of the difference of Weil-Petersson symplectic forms via the Mess map. To perform this computation, the only ingredients we shall need are expressions for the Weil-Petersson symplectic form at an arbitrary point $[\mu] \in \mathcal{T}(\mathbb{D})$ and for the derivative of the Mess map.

Let's start with the description the derivative of the Mess map. We remind the reader, see section 4.3, given a point $([\mu], q) \in T^*\mathcal{T}(\mathbb{D})$, the Mess map associates a pair of harmonic harmonic maps $\Phi_\pm$ with Hopf differentials satisfying

$$\mathrm{iHopf}(\Phi_\pm) = \pm q dw^2. \tag{6.8}$$

We then we obtain the pair $([\mu_+],[\mu_-])$ as the Beltrami coefficients of the composition $\Phi_\pm\circ f_\mu$. To obtain the derivative of this map we shall now compute the variation of the above relation between the quadratic differential q the Hopf differentials $\mathrm{Hopf}(\Phi_\pm)$. Note that we may use the group structure on $\mathcal{T}(\mathbb{D})$ to perform this computation at the the point $[\mu] \in \mathcal{T}(\mathbb{D})$, rather then the preferred point $[0]$. This will bring many simplifications since both Hopf differentials are holomorphic with respect to this conformal structure. Thus we consider the one-parameter families of quasiconformal maps

$$f_{t\delta\tilde{\mu}_\pm} \circ \Phi_\pm(w) = \Phi_\pm(w) + t\delta z_\pm \circ \Phi_\pm(w) + O(t^2),$$

with

$$\partial_{\bar{z}_\pm} \delta z_\pm = \delta\tilde{\mu}_\pm, \qquad \delta\tilde{\mu}_\pm \circ f_{\mu_\pm} = \frac{\delta\mu_\pm}{(1-|\mu_\pm|^2)} \frac{\partial_z f_{\mu_\pm}}{\partial_{\bar{z}} \bar{f}_{\mu_\pm}},$$

and compute the corresponding Hopf differentials

$$\begin{aligned}\frac{4\partial_w\Phi_\pm\partial_w\bar{\Phi}_\pm}{(1-|\Phi_\pm|^2)^2} + t\frac{4|\partial_w\Phi_\pm|^2}{(1-|\Phi_\pm|^2)^2}\Big[\delta\bar{\tilde{\mu}}_\pm \circ \Phi_\pm \frac{\partial_w\Phi_\pm}{\partial_{\bar{w}}\bar{\Phi}_\pm} + \bar{\nu}_\pm^2\delta\tilde{\mu}_\pm \circ \Phi_\pm \frac{\partial_{\bar{w}}\bar{\Phi}_\pm}{\partial_w\Phi_\pm} \\ -(1+|\nu_\pm|^2)\delta\bar{\tilde{\mu}} - 2\bar{\nu}_\pm\partial_w\delta w\Big],\end{aligned} \tag{6.9}$$

where

$$\nu_\pm = \frac{\partial_{\bar{w}}\Phi_\pm}{\partial_w\Phi_\pm} = \pm\mathrm{i}\frac{(1-|\Phi_\pm|^2)^2}{4|\partial_w\Phi_\pm|^2}\bar{q}$$

are the Beltrami differentials of $\Phi_\pm$. Comparing with the variation of q via formula (6.8) we get

$$\delta\bar{\tilde{\mu}}_\pm \circ \Phi_\pm \frac{\partial_w\Phi_\pm}{\partial_{\bar{w}}\bar{\Phi}_\pm} + \bar{\nu}_\pm^2\delta\tilde{\mu}_\pm \circ \Phi_\pm \frac{\partial_{\bar{w}}\bar{\Phi}_\pm}{\partial_w\Phi_\pm} = (1+|\nu_\pm|^2)\delta\bar{\tilde{\mu}} + 2\bar{\nu}_\pm\partial_w\delta w \pm \frac{(1-|\Phi_\pm|^2)^2}{4\mathrm{i}|\partial_w\Phi_\pm|^2}\Big(\delta q + \partial_w q\delta w\Big),$$

which then implies, after some algebra,

$$\begin{aligned}(1-|\nu_\pm|^2)\delta\tilde{\mu}_\pm \circ \Phi_\pm \frac{\partial_{\bar{w}}\bar{\Phi}_\pm}{\partial_w\Phi_\pm} &= (\delta\tilde{\mu} - \nu_\pm^2\delta\bar{\tilde{\mu}}) + 2\nu_\pm\frac{(\partial_{\bar{w}}\delta\bar{w} - |\nu_\pm|^2\partial_w\delta w)}{1+|\nu_\pm|^2} \\ &\mp\frac{(1-|\Phi_\pm|^2)^2}{4\mathrm{i}|\partial_w\Phi_\pm|^2}\frac{\delta\bar{q} + \nu_\pm^2\delta q}{1+|\nu_\pm|^2} \mp \frac{(1-|\Phi_\pm|^2)^2}{4\mathrm{i}|\partial_w\Phi_\pm|^2}\frac{\partial_{\bar{w}}\bar{q}\delta\bar{w} + \nu_\pm^2\partial_w q\delta w}{1+|\nu_\pm|^2}.\end{aligned}$$

This is the expression of the tangent vectors $\delta\mu_\pm \in T_{[\mu_\pm]}\mathcal{T}(\mathbb{D})$ corresponding, via the Mess map, to a vector $(\delta\mu, \delta q) \in T_{([\mu],q)}T^*\mathcal{T}(\mathbb{D})$ written in terms of the holomorphic coordinate associated with $[\mu]$.

6.2.2 Weil-Petersson symplectic form at arbitrary points

Now, we must understand of the Weil-Petersson symplectic form at an arbitrary point of $\mathcal{T}(\mathbb{D})$. The Weil-Petersson symplectic form is defined by

$$\Omega^{\mathcal{T}} = \int_{\mathbb{D}} dz \wedge d\bar{z}\frac{\delta\bar{\mu}\wedge\delta\mu}{(1-|z|^2)^2},$$

at the base point $[0]$. It is then extended to arbitrary points $[\mu_\pm]$ by imposing right invariance under the group structure, that is,

$$\Omega^{\mathcal{T}}_{[0]}(\,\cdot\,,\,\cdot\,) = R^*_{[\mu_\pm]}\Omega^{\mathcal{T}}_{[\mu_\pm]}(\,\cdot\,,\,\cdot\,) = \Omega^{\mathcal{T}}_{[\mu_\pm]}(R_{[\mu_\pm]*}\cdot\,, R_{[\mu_\pm]*}\cdot\,).$$

Such defining property is nothing but a change of integration coordinates and leads to the following expression for the Weil-Petersson symplectic form at an arbitrary point $[\mu_\pm] \in \mathcal{T}(\mathbb{D})$

$$\Omega^{\mathcal{T}}_{[\mu_\pm]} = \int_{f_{\mu_\pm}(\mathbb{D})} dz_\pm \wedge d\bar{z}_\pm \frac{\delta\bar{\tilde{\mu}}_\pm \wedge \delta\tilde{\mu}_\pm}{(1-|z_\pm|^2)^2} = \int_{\mathbb{D}} dz \wedge d\bar{z} \frac{|\partial_z f_{\mu_\pm}|^2}{(1-|f_{\mu_\pm}|^2)^2} \frac{\delta\bar{\mu}_\pm \wedge \delta\mu_\pm}{(1-|\mu_\pm|^2)}$$

and the difference of Weil-Petersson symplectic forms, at point $([\mu_+],[\mu_-]) \in \mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$, now becomes

$$\Omega^{\mathcal{T}}_{[\mu_+]} - \Omega^{\mathcal{T}}_{[\mu_-]} = \int_{\mathbb{D}} dz \wedge d\bar{z} \Big(\frac{|\partial_z f_{\mu_+}|^2}{(1-|f_{\mu_+}|^2)^2} \frac{\delta\bar{\mu}_+ \wedge \delta\mu_+}{(1-|\mu_+|^2)} - \frac{|\partial_z f_{\mu_-}|^2}{(1-|f_{\mu_-}|^2)^2} \frac{\delta\bar{\mu}_- \wedge \delta\mu_-}{(1-|\mu_-|^2)} \Big).$$

We may also use the right invariance property to write this symplectic form as an integral over $f_\mu(\mathbb{D})$

$$\begin{aligned}\Omega^{\mathcal{T}}_{[\mu_+]} - \Omega^{\mathcal{T}}_{[\mu_-]} = \int_{f_\mu(\mathbb{D})} dw \wedge d\bar{w} \frac{|\partial_w \Phi_+|^2 (1-|\nu_+|^2)}{(1-|\Phi_+|^2)^2} \Big[&(\delta\bar{\tilde{\mu}}_+ \circ \Phi_+) \wedge (\delta\tilde{\mu}_+ \circ \Phi_+) \\ &- (\delta\bar{\tilde{\mu}}_- \circ \Phi_-) \wedge (\delta\tilde{\mu}_- \circ \Phi_-) \Big].\end{aligned}$$

Here we have made use of the area preserving properties of $\Phi_- \circ \Phi_+^{-1}$ which, we remind the reader, is minimal Lagrangian, see section 4.2.

A rather tedious computation shows

$$\begin{aligned}(1-|\nu_\pm|^2)(\delta\bar{\tilde{\mu}}_\pm \circ \Phi_\pm) \wedge (\delta\tilde{\mu}_\pm \circ \Phi_\pm) = \frac{(1-|\Phi_\pm|^2)^4}{16|\partial_w \Phi_\pm|^4} \frac{\delta_{(\delta\mu,\delta q)} q \wedge \delta_{(\delta\mu,\delta q)} \bar{q}}{(1+|\nu_\pm|^2)} \\ \pm \frac{(1-|\Phi_\pm|^2)^2}{4\mathrm{i}|\partial_w \Phi_\pm|^2} \Big[\delta_{(\delta\mu,\delta q)} q \wedge \Big(\delta\tilde{\mu} + \frac{2\nu_\pm}{1+|\nu_\pm|^2} \partial_{\bar{w}} \delta\bar{w} \Big) + \delta_{(\delta\mu,\delta q)} \bar{q} \wedge \Big(\delta\bar{\tilde{\mu}} + \frac{2\bar{\nu}_\pm}{1+|\nu_\pm|^2} \partial_w \delta w \Big) \Big] \\ + (1+|\nu_\pm|^2) \Big(\delta\bar{\tilde{\mu}} + \frac{2\bar{\nu}_\pm}{1+|\nu_\pm|^2} \partial_w \delta w \Big) \wedge \Big(\delta\tilde{\mu} + \frac{2\nu_\pm}{1+|\nu_\pm|^2} \partial_{\bar{w}} \delta\bar{w} \Big)\end{aligned}$$

and anti-symmetrizing in $\pm$ we get

$$\begin{aligned}2\mathrm{i} \frac{|\partial_w \Phi_+|^2 (1-|\nu_\pm|^2)}{(1-|\Phi_+|^2)^2} \Big((\delta\bar{\tilde{\mu}}_+ \circ \Phi_+) \wedge (\delta\tilde{\mu}_+ \circ \Phi_+) - (\delta\bar{\tilde{\mu}}_- \circ \Phi_-) \wedge (\delta\tilde{\mu}_- \circ \Phi_-) \Big) = \\ \delta_{(\delta\mu,\delta q)} q \wedge \delta\tilde{\mu} + \delta_{(\delta\mu,\delta q)} \bar{q} \wedge \delta\bar{\tilde{\mu}} + 2q \partial_w \delta w \wedge \delta\tilde{\mu} + 2\bar{q} \partial_{\bar{w}} \delta\bar{w} \wedge \delta\bar{\tilde{\mu}}.\end{aligned}$$

Now, the difference of Weil-Petersson symplectic forms, pulled-back via the Mess map, gives exactly the cotangent bundle symplectic form

$$\begin{aligned}\Omega^{\mathcal{T}}_{[\mu_+]} - \Omega^{\mathcal{T}}_{[\mu_-]} = \int_{f_\mu(\mathbb{D})} dw \wedge d\bar{w} \frac{|\partial_w \Phi_+|^2 (1-|\nu_+|^2)}{(1-|\Phi_+|^2)^2} \Big[&(\delta\bar{\tilde{\mu}}_+ \circ \Phi_+) \wedge (\delta\tilde{\mu}_+ \circ \Phi_+) \\ &- (\delta\bar{\tilde{\mu}}_- \circ \Phi_-) \wedge (\delta\tilde{\mu}_- \circ \Phi_-) \Big] \\ = \frac{1}{2\mathrm{i}} \int_{f_\mu(\mathbb{D})} dw \wedge d\bar{w} \Big(\delta_{(\delta\mu,\delta q)} q \wedge \delta\tilde{\mu} + \delta_{(\delta\mu,\delta q)} \bar{q} \wedge \delta\bar{\tilde{\mu}} + 2q \partial_w \delta w \wedge \delta\tilde{\mu} + 2\bar{q} \partial_{\bar{w}} \delta\bar{w} \wedge \delta\bar{\tilde{\mu}} \Big) = \Omega^{T^*\mathcal{T}}_{([\mu],q)}\end{aligned}$$

which proves, up to the subtleties associated with boundary terms, the Mess map is a symplectomorphism.

6.2.3 Boundary terms and other caveats

We would like to finish this chapter with a few words on the boundary terms dropped in the last computations and the related question of well definedness of the symplectic structures described above.

Already in the context of universal Teichmüller space, the noncompactness of $\mathbb{D}$ introduces subtleties regarding convergence of the symplectic forms. In fact, the universal Weil-Petersson hermitian metric, from which the symplectic form is obtained as the imaginary part, in known diverge unless the harmonic Beltrami coefficients are dual to square integrable holomorphic quadratic differentials [65]. More explicitly, the universal Weil-Petersson hermitian metric is only well defined on

$$\Big\{\delta\mu \in \mathrm{HBD}(\mathbb{D});\, \delta\mu = -\frac{(1-|z|^2)^2}{2}\bar{q} \text{ with } \frac{1}{2\mathrm{i}}\int_{\mathbb{D}} dz\wedge d\bar{z}(1-|z|^2)^2|q(z)|^2 < \infty\Big\}.$$

It should be clear that in the present work we have considered the symplectic forms only at a formal level, not worrying about this convergence problem.

A solution for this problem was introduced by Tahktajan and Teo in [37] with the introduction of a new complex Hilbert manifold structure on $\mathcal{T}(\mathbb{D})$ with well defined Weil-Petersson hermitian metric in each tangent space. We shall not reproduce their arguments here and refer the reader to [37] for more information. We note that it is possible to apply the same arguments to each $\mathcal{T}(\mathbb{D})$ sector of the universal phase space introduced here.

More importantly, we note that the gravitational symplectic form agrees with the difference of Weil-Petersson form only up to boundary terms. Such terms certainly play an important role for the convergence of the symplectic form. We shall not speculate here whether these gravitational boundary terms are enough to make the symplectic form well defined and leave this question for future research.

It should also be noted that in the case of spatially compact AdS spacetimes the arguments presented here are well posed since the symplectic forms under consideration are then completely well defined. In this context, there are still remains some technical issues to be addressed, again regarding boundary terms. We note that when writing the analogous of formula (6.5) for the gravitational symplectic structure for in a spatially compact spacetime manifold we perform the integration over a fundamental domain $\mathbb{D}/\Gamma$. Such fundamental domain is a $4g$-gon with boundaries, described by geodesic on $\mathbb{D}$, being identified by the action of Γ. Therefore, the boundary contributions cannot be discarded on the basis of compactness, but should cancel pairwise via the identification. We shall not present here any demonstration that the needed cancellations do in fact occur, leaving this for future studies. We would just like to describe how such process should be carried out.

We first decompose the fundamental domain's boundary integral as a sum over its geodesic segments

$$\int_{\partial\mathbb{D}/\Gamma} = \sum_{i=1}^{2g}\Big(\int_{c_i} + \int_{-A_i c_i}\Big),$$

where c_i and $A_i c_i$ denote the boundary edges being identified by a generator $A_i \in \Gamma$. Since S is obtained identifying c_i and $A_i c_i$, we need to cancel the unwanted boundary contributions along these geodesics by the use of the Γ invariance properties

$$\mu \circ A \frac{\bar{A}'}{A'} = \mu, \qquad q \circ A (A')^2 = q$$

of Beltrami differentials and holomorphic quadratic differentials.

Note that the definition of the horizontal and vertical sub-bundles of $TT^*\mathcal{T}(\mathbb{D}/\Gamma)$ are not so straightforward anymore. For example, one now needs to make sure the function

$$q + t(\partial_w q \delta w + \delta q),$$

which is was shown holomorphic with respect to $w + \delta w$, also satisfies the correct invariance property under the generators of Γ. Only then it will define a quadratic differential on $\mathbb{D}/\Gamma$. One should then allow, in the definition of the horizontal direction on $T_{([\mu],q)}T^*\mathcal{T}(\mathbb{D}/\Gamma)$, for a more general expression

$$(\delta\mu, \partial_w q \delta w + Q) \in H_{([\mu],q)}T^*\mathcal{T}(\mathbb{D}/\Gamma),$$

where Q satisfies the transformation

$$Q \circ A(A')^2 = Q + 2\partial_w q \Big(\delta w - \delta w \circ A \frac{1}{A'}\Big) + 4q\partial_w \Big(\delta w - \delta w \circ A \frac{1}{A'}\Big) + 2q(\partial_w \delta w) \circ A.$$

Only after solving such equation it is possible to describe the horizontal space. This is a technical difficulty we shall leave for future work

Chapter 7

Conclusion

This final chapter concludes with a summary of achieved results and some possible future research directions.

7.1 Conclusion

7.1.1 Results and future directions

In this thesis, we have provided two equivalent parametrizations of what can be called the universal phase space of 2+1 AdS spacetimes. These were given in terms of two copies $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ of universal Teichmüller space and by the cotangent bundle $T^*\mathcal{T}(\mathbb{D})$ over that space. The use of the adjective "universal" is the same as in universal Teichmüller theory, since our phase space contains the phase spaces of all spatially compact globally hyperbolic AdS spacetimes, as well as that of all multi black holes, as submanifolds. It also contains the Brown-Henneaux asymptotic excitations on the conformal boundary.

While the standard description of asymptotically AdS spacetimes is given by deformations of, say, AdS_3 by non-trivial asymptotic symmetries generated by Brown-Henneaux vector fields, we have described these spacetimes by considering two types of deformations of the domain of dependence of a spacelike geodesic surface in AdS_3. The first, interpreted as "geometric" direction, was obtained by prescribing a deformation of the boundary curve of the surface, determining a new maximal surface in AdS_3. This corresponds to the "difference" between the two Teichmüller sectors in $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$, and to the vertical direction in $T^*\mathcal{T}(\mathbb{D})$. The second, was given by a purely spatial quasiconformal deformation applied to the initial spacelike geodesic surface. This did not alter the intrinsic or extrinsic properties of maximal surfaces in AdS_3 and, therefore, was interpreted as a "non-geometric" direction. It corresponds to the "sum" of the Teichmüller sectors in $\mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$, and to the horizontal direction in $T^*\mathcal{T}(\mathbb{D})$.

Our construction was based on results already known in the mathematics literature, mainly

[39, 40, 41], where different geometric aspects of maximal, or more generally constant mean curvature, surfaces in AdS_3 were described in terms of universal Teichmüller theory. We have extended these results to the context of 2+1 AdS spacetimes where, from the work of Brown and Henneaux [26], it is clear one also needs to account for non-geometric data associated with non-trivial asymptotic symmetries. Therefore, the main contribution in this thesis comes in the realization of these asymptotic symmetries in terms of maximal surface data in AdS_3 which, in fact, turns out to be a quite natural generalization of the compact case phase space of [15] and [13], see also [14] for the relation of these descriptions in terms of maximal surfaces.

On the other hand, in the physics literature, most attention was given to the non-geometric (asymptotic) aspects of the theory, mainly due to its relations to Maldacena's AdS/CFT conjecture [28]. The phase space of 2+1 AdS gravity, from this point of view, was described by two copies of the quotient $\mathrm{Diff}_+(\mathbb{S}^1)/SL(2,\mathbb{R})$, see e.g. [30], but the spacetime bulk moduli, parametrizing nontrivial spatial topology, were not available. By expressing the asymptotic symmetries in terms of maximal surface data, our results now extend this description to include the bulk moduli and thus the topological information about the whole spacetime.

More explicitly, our work gives a new interpretation for the Brown-Henneaux generators in terms of Zygmund class vector fields on $\mathbb{S}^1$. This was obtained from an infinitesimal analysis, around AdS_3, of the spacetime metrics arising in our maximal surface parametrization and in the, more standard, holographic Fefferman-Graham type parametrization of spatially-noncompact AdS spacetimes. We have described infinitesimal deformations of the AdS_3 metric in the direction of both quasiconformal and Brown-Henneaux generators, and were then able to match these generators asymptotically by imposing the resulting spacetimes to represent the same physical state. This shows, at the infinitesimal level, there is a well defined analytic continuation of the spacetime metric on the maximal surface's domain of dependence beyond the Cauchy horizon, to a region including the components of the conformal boundary. It is then expected that such continuation can be extended to the case of finite transformations thus identifying, to a certain extent, the group of asymptotic symmetries of AdS spacetimes and the quasiconformal deformation space of domains of dependence of maximal surfaces in those spacetimes.

Note that such an identification cannot be one-to-one since, although enough to describes all possible asymptotically AdS metrics in a neighbourhood of conformal infinity, the group of asymptotic symmetries does not contain the bulk geometry moduli, that is, they do not fix the internal spacetime topology. This is the main advantage of the new parametrization proposed in this thesis. The maximal surface data also provides the bulk moduli parameters via the Γ invariance properties of the associate pair of quasisymmetric homeomorphisms of the unit circle. Thus, if $\phi_\pm \in \mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ are Γ invariant, in the sense that $A \circ \phi_\pm \circ A^{-1} \in \mathrm{Möb}(\mathbb{S}^1)$ for all $A \in \Gamma$, we may take the quotient of the arising AdS metric on $\mathbb{R} \times \mathbb{D}$ to a metric in $\mathbb{R} \times \mathbb{D}/\Gamma$.

Then, on each boundary component of $\mathbb{D}/\Gamma$ we have pairs of homeomorphisms into $\mathbb{S}^1$, and these are the moduli parametrizing the group of asymptotic symmetries.

This now suggests a possible application of our construction to the debate on the microscopic origin of 2+1 dimensional black hole entropy. Since the universal space includes all possible multi black holes, together with the Brown-Henneaux excitations in each of their asymptotic regions, it seems that one could now compute the partition function with, say, fixed temperature and angular velocity on a single asymptotic region, by summing over all inner horizon geometries. In terms of the model A realization, one would then fix some properties of the quasisymmetric homeomorphisms $\phi_\pm$ on some appropriate interval $I \subset \mathbb{S}^1$, say determined by the fixed points of a fixed hyperbolic-hyperbolic generator, and sum over all quasisymmetric homeomorphisms satisfying those properties. Note this would also include a sum over topologies coming from Γ invariance properties of the restriction of $\phi_\pm$ to the complement of I. The entropy could then be extracted from this canonical partition function by the standard thermodynamic formulas. Note that for such computation to work one needs expressions for the spacetime charges in terms of the quasisymmetric homeomorphisms $\phi_\pm$. Our (infinitesimal) expression for the charges is then a first step in this direction.

Another interesting future direction would be to reformulate such partition function computation as that in the context of some conformal field theory. In this respect we note that the Gauss-Codazzi equations that arise on the maximal surface description are integrable, associated with the so-called sl_2 conformal affine Toda system [70]. It might be that the conformal field theory corresponding to this system is of relevance for the quantum description of AdS_3 gravity. We note that such conformal field theory would naturally live on the maximal surface, not on the asymptotic boundary. On the other hand, we have seen that the analytic continuation (to the imaginary time) of the chiral functions on $\partial_\infty \mathrm{AdS}_3$ has a natural interpretation in terms of maximal surface's data. At least at the infinitesimal level, the descriptions of $T\mathcal{T}(\mathbb{D})$ obtained in [65] can be used to show the variations of quasilocal stress-tensor of Brown and York are closely related to the B model realization of $\mathcal{T}(\mathbb{D})$. This then admits an immediate generalization to the finite case, with the components of the (analytic continued negative frequency part of the) quasilocal stress-tensor being given by the Bers embedding holomorphic/anti-holomorphic quadratic differentials. Giving a proof that such finite relation is indeed realized in this manner is an important open problem we would like to address in future works. Nonetheless, already the infinitesimal relation suggests it might be possible to translate Lorentzian CFTs on the conformal timelike boundary to Euclidean CFTs on the spacelike maximal surface, and vice versa. It is thus possible that Euclidean signature CFTs, in particular the sl_2 Toda field theory, are of interest for the AdS/CFT type description of 2+1 dimensional quantum gravity.

Finally, it would also be interesting to investigate directly the quantization of the universal phase space of AdS gravity and its relation to the theory of quantum Teichmüller spaces. These were introduced by Kashaev [16] and, independently, Chekhov and Fock [17] in the setting of punctured Riemann surfaces. Generalizations for the universal Teichmüller space context were also discussed in [18]. We have presented, in this thesis, an analysis of the symplectic properties of the map Mess : $T^*\mathcal{T}(\mathbb{D}) \to \mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$. We have seen that, at a formal level, the bulk contributions of the symplectic form induced by the gravitational action agrees with the natural symplectic forms on both parametrizations of the phase space. This not only shows that AdS 2+1 quantum gravity could be understood within the framework of quantum Teichmüller theory, but may also lead to new ways of addressing problems in this theory by recasting the Weil-Petersson form (or rather the difference of two such symplectic forms) as a much simpler cotangent bundle symplectic form.

This was obtained by writing the gravitational pre-symplectic form, in both the ADM and Chern-Simons formulations, directly in terms of the reduced phase space variables, $([\mu], q) \in T^*\mathcal{T}(\mathbb{D})$ and $([\mu_+], [\mu_-]) \in \mathcal{T}(\mathbb{D}) \times \mathcal{T}(\mathbb{D})$ respectively. In the case of the Weil-Petersson symplectic form, coming from the Chern-Simons formulation, a more detailed analysis was needed. We thus explained how to describe this symplectic form at an arbitrary point in $\mathcal{T}(\mathbb{D})$ by making use of the right group structure induced by composition of quasiconformal maps. Together with a description of the derivative of the Mess map, this was enough to pull-back the difference of Weil-Petersson symplectic forms at $[\mu_+]$ and $[\mu_-]$ to a symplectic form in $T^*\mathcal{T}(\mathbb{D})$, which agreed with the cotangent bundle symplectic form.

As a final remark, we note that although our arguments in the last chapter still need some improvements, in particular regarding the boundary contributions, they represent an important first step in the direction of a proof that the Mess map is symplectic. This is non-trivial even in the case of compact spatial topologies, where the problem of well definiteness of the symplectic structure is not present. Work is in progress on completing this proof.

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